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Brane-world inflation and the transition to standard cosmology

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Abstract: In the context of a five-dimensional brane-world model motivated from heterotic M-theory, we develop a framework for potential-driven brane-world inflation. Specifically this involves a classification of the various background solutions of $(A)dS_5$ type, an analysis of five-dimensional slow-roll conditions and a study of how a transition to the flat vacuum state can be realized. It is shown that solutions with bulk potential and both brane potentials positive exist but are always non-separating and have a non-static orbifold. It turns out that, for this class of backgrounds, a transition to the flat vacuum state during inflation is effectively prevented by the rapidly expanding orbifold. We demonstrate that such a transition can be realized for solutions where one boundary potential is negative. For this case, we present two concrete inflationary models which exhibit the transition explicitly.

Keywords: Cosmology of Theories beyond the SM, M-Theory, Extra Large Dimensions.

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Contents

1. Introduction

Over the past couple of years there has been much interest in the cosmological consequences of models in which the standard model of particle physics is located on a brane embedded in a higher-dimensional bulk. So far, the most popular of these models have been ones in which the universe has topology $\mathcal{M}_4 \times \mathcal{S}^1/\mathbb{Z}_2$ motivated by Hořava-Witten theory [1, 2, 3] compactified on a Calabi-Yau 3-fold [4, 5, 6]. To achieve gauge-unification at $\sim 10^{16}\,\mathrm{GeV}$ and a Planck scale of $\sim 10^{19}\,\mathrm{GeV}$, this theory requires that the orbifold radius is roughly two orders of magnitude larger than the length scale of the Calabi-Yau [3]. Furthermore, the orbifold is currently at an energy scale which may have been accessible during inflation. If the scale of extra dimensions is indeed $\sim 10^{14}\,\mathrm{GeV}$ or higher, ground-based accelerators seem to have little prospect of directly observing their effects in the forseeable future. Therefore there is great interest in looking for cosmological signatures that are likely to arise from the different evolution in the five- and four-dimensional regimes.

For the reason mentioned above, inflation seemed a natural place to begin the investigation. The first steps were taken in ref. [7] where explicit five-dimensional background solutions were derived for the case of an empty bulk, with the branes each supporting scalar potentials. More general solutions including bulk potentials were found in e.g. refs. [8]–[17]. These solutions couple to constant bulk and brane potentials and, therefore, relate to realistic brane-world inflation in much the same way four-dimensional de Sitter space relates to four-dimensional slow-roll inflation.

The main goal of the present paper is to develop a more realistic framework for brane-world inflation. Our procedure to do this will be analogous to the one employed in the four-dimensional context, where three major steps are involved. It may be helpful in the following to briefly recall what these steps are in the familiar four-dimensional setting. First, one needs to find a cosmological solution which couples to a constant positive potential. In four dimensions, the only possibility is dS_4, of course. Secondly, one needs to find approximate solutions "near" dS_4 with a slowly varying scalar field potential, in short, one needs to analyze the slow-roll approximation. Thirdly, one needs to find a way to end inflation and the theory needs to relax into its vacuum state corresponding to a vanishing scalar potential. While this is quite familiar and straightforward in four dimensions, the analogous steps in five dimensions are far less trivial, as we will see.

Our analysis will be performed in the context of the five-dimensional brane-world model first introduced in ref. [7]. As mentioned, this model is based on the orbifold S^1/Z_2 and is motivated by Calabi-Yau compactifications of heterotic M-theory [4, 5, 6]. In the simple version to be used in this paper, the model has a single bulk scalar field ϕ with bulk potential $V(\phi)$ and one scalar ϕ_i with potential

$V_i(\phi, \phi_i)$ on each of the two boundaries labeled by $i = 1, 2$. Note that we generally take the boundary potential V_i to depend on the bulk scalar ϕ, as well as on the boundary scalar ϕ_i.

For any theory involving branes, it is always possible to geometrically project the full theory onto one of the branes. An observer confined to the brane who sees such a projected theory may notice many interesting phenomena, such as non-standard cosmology [10] and even antigravity [33]. These may all be viewed as consequences of information from the bulk influencing the behaviour of the brane, and in this sense the projected theory is still aware of the higher-dimensional theory. If a full description of the higher-dimensional dynamics is known, then an exact solution of the projected theory may always be obtained by restricting the complete solution to the appropriate brane. At low energies however, there will also be a description in terms of four-dimensional effective theories, valid when all higher Kaluza-Klein cease to play a role. Such a theory will be truly four-dimensional in the sense that it possesses a vacuum state with four-dimensional Poincaré symmetry and excitations about this vacuum satisfy the usual four-dimensional Einstein equations and other field equations.

It is these effective four-dimensional theories that we expect to describe the evolution of the late universe and which are vital to extract concrete predictions from brane-world models. For simplicity we restrict ourselves to vacuum states which couple to constant potentials V, V_i. Then there are two types of vacua which arise for specific values of the potentials, namely five-dimensional Minkowski space for $V = V_1 = V_2 = 0$ and the warped vacuum due to Randall and Sundrum [18] for $V_1 = -V_2 = \pm\sqrt{12|V|}$. In this paper, for simplicity, we will be mostly concerned with the flat Minkowski vacuum. This also seems appropriate in the M-theory context since the flat vacuum can be viewed as an approximation to the domain wall vacuum of heterotic M-theory [5]. However, most of the methods and ideas presented in this paper should analogously apply to warped vacuum states.

Let us now return to the three steps towards a realistic framework for inflation mentioned above and discuss them separately in the context of our brane-world model. First we need to find cosmological solutions which couple to constant potentials V, V_i. Clearly there is a variety of possibilities and, hence, the situation is far more complicated than in four dimensions. In this paper, we will focus on solutions which are part of dS_5 or AdS_5, for simplicity. Again, we expect that our ideas can analogously be applied to the Schwarzschild-$(A)dS_5$ solutions found in ref. [16, 11] and it would be interesting to carry this out explicitly. We will systematically classify the solutions of $(A)dS_5$ type particularly in relation to a number of properties essential for our subsequent discussion, such as the existence of solutions with static orbifold and the allowed ranges for the potentials V, V_i.

Secondly, we will be implementing slow-roll of the various scalar fields for the most interesting of those solutions. While there are some discussions of brane-world slow-roll in the literature [19, 20] they generally focus on slow-roll on a single brane.

Here, we will consider the the full five-dimensional background where we allow for slow-roll of both boundary scalars as well as the bulk scalar. Particularly the latter, being a five-dimensional field, complicates matters significantly and makes the analysis much more involved than it is in four dimensions.

Finally, we need to understand how to end inflation and approach the (flat) vacuum state in order to make contact with the four-dimensional effective theory that governs the late evolution of the universe. In this context, it is perhaps worth pointing out that the background solutions under discussion do not have an effective four-dimensional description for most values of the potentials V,V_i. That is to say, generically, they cannot be understood as solutions of an effective four-dimensional theory which involves a finite number of (zero mode) fields. The reason for this is of course the existence of the brane potentials V_i which constitute sources localized in the orbifold and which excite Kaluza-Klein modes in a non-thermal, coherent way. Clearly, unlike thermal excitations, these coherent modes are not diluted away by inflation. It is only if we go to a region of solution space where the potentials V_i (and, hence, the coherent Kaluza-Klein excitations) are sufficiently small (in a sense to be made precise later) that we can hope for an effective four-dimensional description. All this is in contrast to inflation in a more traditional Kaluza-Klein setting without branes. There, the Kaluza-Klein modes can always be set to zero consistently and the dynamics is governed by an effective four-dimensional theory for the zero modes.

Of course, the early universe is not well enough understood to be able to assert that it must look four-dimensional by the end of inflation. A conservative estimate would say that nucleosynthesis [21, 22] is the earliest time at which we may be sure standard cosmology applies. However, in part motivated by the energy scales of Hořava-Witten theory, in this paper we will indeed look for inflationary solutions that do evolve from a five- to a four-dimensional regime during inflation. This has the advantage of allowing us to follow an explicit cosmological solution to the full five-dimensional theory as it evolves, and then match it on to standard cosmology once a 4D description applies. It is only when one has such an explicit background that it is possible to apply the formalism developed in refs. [23]–[26] to describe the evolution of cosmological fluctuations during the five-dimensional era and predict how they would appear in the CMB today.

Our search for such backgrounds will proceed in two steps. Based on the classification of solutions, we will first analyze how the properties of a given class change as we vary the constant potentials V and V_i "by hand". This way, we single out certain classes which are generically five-dimensional but develop an effective four-dimensional description in certain regions of the V, V_i parameter space. For those regions, we will determine the corresponding solution to the four-dimensional effective action. Of course, since the universe is dynamic, the mere existence of such regions of parameter space is insufficient. For the solutions to be realistic, it is also necessary that they actually evolve towards this region. Since we wish to know whether the background solutions dynamically evolve towards a four-dimensional description, it is obviously not appropriate to study slow-roll within a four-dimensional context from the start. We will therefore apply our results for five-dimensional slow-roll to single out appropriate backgrounds. As an additional complication, it is necessary to ensure that slow-roll does not attempt to take the full solution out of the region of parameter space for which it is valid.

Let us now summarize our main results. Our classification of $(A)dS_5$ background solutions leads us to consider four different types of solutions specified by the signs of V and $V_1 + V_2$ which turn out to have quite different properties. We explicitly determine all solutions for these four classes. Most notably, we show that for the case $V > 0$ and $V_1 + V_2 > 0$ no non-singular static orbifold solution or separating solution exists. However, we do find non-separating solutions with dynamical orbifold in this case. To our knowledge, these are the first $(A)dS_5$ solutions which allow one to have all potentials positive, that is $V > 0$ and $V_i > 0$ where $i = 1, 2$. Under certain additional assumptions, we show that similar statements hold for the case $V < 0$, $V_1 + V_2 > 0$. Another general results concerns the ratios $\rho_i \equiv V_i/\sqrt{12|V|}$. We show that solutions with $V < 0$ only exist if the boundary potentials satisfy the additional constraints $|\rho_i| \geq 1$ and $V_1 V_2 < 0$. No such constraints apply to the case $V > 0$ where V_1 and V_2 can be chosen arbitrarily.

Which of these solutions exhibit a four-dimensional limit with respect to the flat vacuum state? As a general rule of thumb, we find that such a limit can only be realized in the region of parameter space characterized by $|\rho_i| \ll 1$. This basically excludes all solutions with $V < 0$ as they require $|\rho_i| \geq 1$. Further, for solutions with $V > 0$, being in the region where $|\rho_i| \ll 1$ is a sufficient condition for exhibiting a $D = 4$ limit as long as the orbifold is static. For a dynamical orbifold, on the other hand, having $|\rho_i| \ll 1$ often leads to a four-dimensional limit only within a certain time range. This is precisely what happens for solutions with $V > 0$ and $V_1 + V_2 > 0$ which, as we have mentioned, always have a dynamical orbifold. However, within the class $V > 0$ and $V_1 + V_2 < 0$, we do find examples exhibiting a $D = 4$ limit whenever $|\rho_i| \ll 1$. For each of these solutions, we examine the regime where they possess a four-dimensional description, and explicitly calculate the four-dimensional metric. For static orbifold cases, the four-dimensional limit is simply dS_4 while for dynamical orbifold we typically find power-law expansion.

In order to study how these models evolve during inflation, we have undertaken a thorough investigation of the slow-roll conditions that should be imposed on the fields to make the background solutions consistent. For the bulk scalar field ϕ, this task is significantly more complicated than in the standard four-dimensional case. This is partly because the form of the background metrics means it is impossible to consider ϕ as independent of the orbifold co-ordinate and partly because there are boundary conditions (analogous to the Israel conditions on the metric) that ϕ must satisfy at each of the branes. We have found the appropriate slow-roll equation for ϕ

and its general bulk solution. The form of this is somewhat complicated and hence we find explicit solutions to the boundary conditions in two cases; when the bulk and brane potentials are related by $V_i = \mathrm{U}_i(\phi_i)[V(\phi)]^{1/2}$ and for arbitrary potentials where the background metric has a static orbifold radius.

Finally, we have combined the above results to study the dynamical transition from five to four dimensions. From our statements above it is clear that this involves having the scalar fields slow-roll such that the quantities $|\rho_i|$ evolve from values $|\rho_i| \geq 1$ at the beginning of inflation to $|\rho_i| \ll 1$ at the end. Seemingly, the solution with all potentials positive is the most attractive to dynamically realize such an evolution. We find that this solutions indeed rolls towards the Minkowski vacuum. Unfortunately, because of the expanding orbifold, the solution becomes again five-dimensional after a certain critical time corresponding to a few e-folds, at most. Therefore, for this solution, most of inflation takes place in five-dimensions and the transition to $D = 4$ must happen during either (p)reheating or radiation domination. At present we do not possess explicit solutions to the five-dimensional equations during these epochs, and so for now we must set this solution to one side.

This leaves us with the solutions for $V > 0$ and $V_1 + V_2 < 0$. At first, it seems impossible that slow-roll allows us to evolve close to the Minkowski vacuum at all in those cases — how can the negative brane potential which is required roll "uphill"? We show that this can happen so long as the brane potential depends on the bulk scalar field ϕ more strongly than $\sqrt{V(\phi)}$ and less strongly than $V(\phi)^2$. If this is true, then there is no objection to a four-dimensional description at the end of inflation. This leads to one of the main results of this paper — we present a background solution that undergoes arbitrarily many e-folds of five-dimensional inflation and is slow-rolling to a four-dimensional description before inflation ends. However, we also point out that there remains a problem with the stability of the background solution which is closely related to the problem of how to stabilize the orbifold. As a concrete application, we demonstrate how the transition occurs for two simple choices of bulk scalar potential, a "new inflation" model with $V(\phi) = M^2 - \frac{1}{2}m^2\phi^2 + \cdots$ and a "chaotic" model with $V(\phi) = \frac{1}{2}m^2\phi^2$. The purpose here is not to claim that these potentials are realistic, but simply to show explicitly how the transition to four dimensions occurs. Interestingly, for the two potentials, the transition occurs at different rates, and at different stages of inflation. It is conceivable that still different behaviour would be exhibited by other potentials, such as those inspired by supersymmetry [27].

After this transition, the solutions behave as either dS_4 or power-law inflation, depending on the behaviour of the orbifold radius. For the first time, this solution allows one to make contact between the physics of perturbations in brane-world models and standard four-dimensional cosmology.

The paper is organized as follows. In section 2 we review the five-dimensional action which constitutes the starting point for our investigation. This may be familiar to many readers and is included for completeness. Section 2.2 outlines the

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procedure for obtaining both the four-dimensional theory based on the Minkowski vacuum and the general conditions for a five-dimensional solution to have an effective four-dimensional limit. In section 3 we categorize all the five-dimensional solutions with $(A)dS_5$ bulk geometry according to the signs of their potentials. The calculations underlying this categorization are somewhat technical and can be found in appendices A, B and C. Next, we focus our attention on the solutions with $V > 0$ and derive their four-dimensional limits in accordance with the general theory of section 2.2. In section 4 our attention shifts to establishing the general slow-roll conditions. Section 5 then discusses how each of the $V > 0$ solutions evolves under slow-roll and whether it is possible for them to achieve ~ 65 e-folds of inflation combined with a transition to $D = 4$. Finally, in section 6 we study the behaviour of a certain class of solutions that has met all the criteria, showing explicitly how the four-dimensional transition occurs. This final section is somewhat self-contained and includes a brief review of the main results of the rest of the paper for those readers who may be more interested in brane-world phenomenology then technical details.

2. The theories

This section will briefly review the five-dimensional theory, first proposed in ref. [7], that forms the basis for our investigation of cosmology in the presence of branes. After specifying the vacuum states of this theory, we derive the effective four-dimensional theory associated with the flat vacuum state. This effective action forms the low energy limit of the theory which one expects to govern the late "standard" evolution of the universe.

2.1 The five-dimensional action

The action for our five-dimensional theory is given by

$$S_5 = -\frac{1}{2\kappa_5^2}\left[\int_{\mathcal{M}_5}\sqrt{-g}\left[R+\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi+V(\phi)\right]+ + \sum_{i=1}^{2}\int_{\mathcal{M}_4^i}\sqrt{-g}\left[\frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i+V_i(\phi_i,\phi_i)\right]\right]. \qquad (2.1)$$

Co-ordinates x^α with indices $\alpha, \beta, \gamma, \ldots = 0, \ldots, 3, 5$ label the bulk space $\mathcal{M}_5$. The fifth dimension is compactified on a circle with coordinates $y \equiv x^5 \in [-R, R]$ and the endpoints of the interval identified. As usual, the orbifold S^1/Z_2 is obtained by implementing the Z_2 action $y \to -y$. The branes $\mathcal{M}_4^i$, $i = 1, 2$ are located at the fixed points $y = y_i =$ const of the orbifold, where $y_1 = 0$ and $y_2 = R$. Four-dimensional coordinates longitudinal to the brane will be denoted by x^μ with $\mu, \nu, \ldots = 0, \ldots, 3$. With our normalization, the brane scalar fields and potentials have mass dimensions $-1/2$ and 1, respectively, whereas the bulk scalar fields and potentials have mass dimensions 0 and 2. The above action is motivated by five-dimensional heterotic

M-theory [4] and is designed to capture the main properties of this theory which are essential for cosmology. Within heterotic M-theory, the potentials V and V_i have certain universal parts depending on the dilaton which we have not made explicit, for simplicity. Also, there are further perturbative contributions to V_i which are, in principle calculable but rather model-dependent. Some progress has been made in ref. [28, 29, 30] towards understanding the bulk potential V which, apart from its universal piece, is of non-perturbative origin. In view of the model-dependence of all these considerations, here we will view V and V_i as arbitrary, and look to specify what properties they should have in order to be cosmologically viable. Notice in particular that the brane scalar field potential may in general depend on both the brane and bulk scalar fields.

We take the most general Ansatz for the metric consistent with a homogeneous, isotropic, three-dimensional universe with the additional simplification of spatial flatness as we expect to be appropriate during inflation.[1] Our metric Ansatz is then given by

$$ds^2_{(5)} = -e^{2\nu(\tau,y)}d\tau^2 + e^{2\alpha(\tau,y)}\delta_{ij}dx^i dx^j + e^{2\beta(\tau,y)}dy^2 \,. \tag{2.2}$$

Using this Ansatz while varying the above action leads to the following Einstein equations

$$3e^{-2\nu}(\dot\alpha^2 + \dot\alpha\dot\beta) - 3e^{-2\beta}(\alpha'' - \alpha'\beta' + 2\alpha'^2) = \frac{1}{4}e^{-2\nu}\dot\phi^2 + \frac{1}{4}e^{-2\beta}\phi'^2 + \frac{1}{2}V \,, \tag{2.3}$$

$$\begin{aligned}&e^{-2\nu}(2\ddot\alpha + \ddot\beta + (3\dot\alpha + 2\dot\beta - 2\dot\nu)\dot\alpha + (\dot\beta - \dot\nu)\dot\beta) - \\ &- e^{-2\beta}(2\alpha'' + \nu'' + (3\alpha' - 2\beta' + 2\nu')\alpha' + (\nu' - \beta')\nu') = -\frac{1}{4}e^{-2\nu}\dot\phi^2 + \frac{1}{4}e^{-2\beta}\phi'^2 + \frac{1}{2}V \,,\end{aligned} \tag{2.4}$$

$$3e^{-2\nu}(\ddot\alpha - \dot\nu\dot\alpha + 2\dot\alpha^2) - 3e^{-2\beta}(\alpha'^2 + \nu'\alpha') = -\frac{1}{4}e^{-2\nu}\dot\phi^2 - \frac{1}{4}e^{-2\beta}\phi'^2 + \frac{1}{2}V \,, \tag{2.5}$$

$$(\dot\alpha' + \dot\alpha\alpha' - \dot\alpha\nu' - \dot\beta\alpha') = -\frac{1}{2}\dot\phi\phi' \tag{2.6}$$

and scalar field equations in the boundary picture

$$e^{-2\nu}(\ddot\phi + (3\dot\alpha + \dot\beta - \dot\nu)\dot\phi) - e^{-2\beta}(\phi'' + (3\alpha' - \beta' + \nu')\phi') = -\partial_\phi V \,, \tag{2.7}$$

$$\ddot\phi_i + (3\dot\alpha - \dot\nu)\dot\phi_i = -e^{2\nu}\partial_{\phi_i}V_i \,. \tag{2.8}$$

Here a dot (prime) denotes differentiation with respect to τ (y). These bulk equations are subject to the boundary conditions

$$e^{-\beta}\alpha' \,|_{y=y_i} = \mp\frac{1}{12}\left[\frac{1}{2}e^{-2\nu}\dot\phi_i^2 + V_i\right]_{y=y_i} \,, \tag{2.9}$$

$$e^{-\beta}\nu' \,|_{y=y_i} = \mp\frac{1}{12}\left[-\frac{5}{2}e^{-2\nu}\dot\phi_i^2 + V_i\right]_{y=y_i} \,, \tag{2.10}$$

$$e^{-\beta}\phi' \,|_{y=y_i} = \pm\frac{1}{2}\left[\partial_\phi V_i\right]_{y=y_i} \,, \tag{2.11}$$

[1] For certain solutions it will be trivial to drop this restriction.

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where the upper (lower) sign applies at the first (second) brane located at $y = 0$ ($y = R$). These conditions describe permissible discontinuities in the gradient of the metric and bulk scalar field in the y-direction caused by their coupling to the energy-momentum tensors and fields on the branes.

The five-dimensional action (2.1) is well known to admit two types of vacuum states which couple to constant potentials V, V_i and respect four-dimensional Poincaré invariance. The first is simply five-dimensional Minkowski space, which requires $V = V_1 = V_2 = 0$, and the second is the Randall-Sundrum vacuum [18] with metric

$$ds^2_{(5)} = e^{-2ky}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2\,, \tag{2.12}$$

where $k = \sqrt{-V/12}$, which is a solutions provided $V_1 = -V_2 = \pm\sqrt{-12V}$ with our normalisation of the potentials. Near each of these vacua we have a four-dimensional effective theory which describes the zero-mode dynamics of the respective vacuum state. This four-dimensional description is typically valid as long as all four-dimensional momenta are smaller than the mass of the first Kaluza-Klein mode of the vacuum state, so that the excitation of Kaluza-Klein modes is negligible. In this paper, we focus on the flat vacuum state which can also be viewed as an approximation to the domain-wall vacuum of five-dimensional heterotic M-theory. However, many of the concepts presented in this paper should analogously apply to the warped vacua of ref. [18], although handling the details will be more complicated. It would be interesting to work this out explicitly.

From (2.9) and (2.10) it is clear that any solution of the five-dimensional equations of motion will be inhomogeneous in the orbifold direction whenever the branes' energy-momentum tensors are non-negligible. This inhomogeneity corresponds to a coherent excitation of Kaluza-Klein modes. Therefore, from a five-dimensional viewpoint, smallness of the brane energy-momentum tensors is a necessary condition for Kaluza-Klein modes to be negligible and, hence, for the validity of the $D = 4$ effective theory.

2.2 The four-dimensional low-energy description

Let us now review the four-dimensional effective action associated with the flat vacuum state. This low-energy description should be distinguished from the apparent four-dimensional view one arrives at by projecting the full theory onto one of the orbifold fixed planes. In the latter picture, an observer on the brane may perceive many non-conventional phenomena, including coupling to gravity via the five-dimensional Newton constant κ_5^2, non-local effects from the bulk encoded in the projected Weyl tensor $E_{\mu\nu}$ and even antigravity on a negative tension brane [31, 32, 33]. These effects are a consequence of the brane being embedded in a higher-dimensional bulk, a situation which can obviously influence the physics on the brane. In this paper, we will be working with a field content that is simple enough to provide explicit solutions to the full five-dimensional theory. By projecting these solutions onto one of the branes, it is in principle possible to recover all the effects discussed above (although actually no non-local effects will be present because of the high degree of symmetry of our solutions). If more complicated evolution were being considered, in general it is an extremely difficult task to find complete bulk solutions, and then ones' best hope is to attempt to infer various properties of the theory by looking at one of the branes. Since it is simply a geometrical projection, the theory on the branes may always be found, but the presence of the effects mentioned above means that it does not obey, for example, standard four-dimensional Einstein gravity.

Exactly as with a standard Kaluza-Klein theory, in certain cases it may happen that the five-dimensional solution evolves in such a way as to make all higher Kaluza-Klein modes negligable. If and only if this is the case, a low-energy four-dimensional description becomes permissable. It is this low-energy theory, together with the conditions for its validity, that we wish to discuss in this section.

As always in the presence of branes, and unlike a pure Kaulza-Klein theory, the KK modes around the vacuum cannot be set exactly to zero in a consistent way. However, under the conditions stated in the previous section, they constitute small excitations which can be neglected. It is then appropriate to perform a reduction to $D = 4$ using the vacuum solution

$$\begin{aligned} ds^2 &= \bar{g}_{\mu\nu} dx^\mu dx^\nu + e^{2\bar{\beta}} dy^2 \, , \\ \phi &= \bar{\phi} \end{aligned} \tag{2.13}$$

with the slowly-varying collective modes $\bar{\beta} = \bar{\beta}(x^\mu)$, $\bar{g}_{\mu\nu} = \bar{g}_{\mu\nu}(x^\mu)$ and $\bar{\phi} = \bar{\phi}(x^\mu)$ inserted. Defining the four-dimensional fields

$$g^{(4)}_{\mu\nu} = e^{-\bar{\beta}} \bar{g}_{\mu\nu} \, , \qquad T = e^{\bar{\beta}} \, , \qquad S = e^{\bar{\phi}} \, , \qquad C_i \equiv \sqrt{\frac{M_5^3}{3}} \phi_i \tag{2.14}$$

one finds for the effective action

$$\begin{aligned} S_{(4)} &= -\frac{1}{16\pi G_N} \int_{\mathcal{M}_4} \sqrt{-g_4} \left[R_4 + \frac{3}{2} T^{-2} \partial_\mu T \partial^\mu T + \frac{1}{2} S^{-2} \partial_\mu S \partial^\mu S \right] - \\ &\quad - \int_{\mathcal{M}_4} \sqrt{-g_4} \left[\frac{1}{2} \sum_{i=1}^{2} K_i \partial_\mu C_i \partial^\mu C_i + V_4 \right] \end{aligned} \tag{2.15}$$

with the four-dimensional potential

$$V_4 = \frac{1}{2T^2} \sum_{i=1}^{2} V_{4i} + \frac{1}{T} R M_5^3 V \, . \tag{2.16}$$

Here the boundary potentials V_{4i}, normalized to four mass dimensions, are defined by $V_{4i} = M_5^3 V_i$ and the four-dimensional Newton constant is $G_N = 1/16\pi R M_5^3$. The "Kähler metrics" K_i are given by $K_i = 3/2T$. The above action is a lowest-order expression and receives corrections from small but non-vanishing Kaluza-Klein modes which have to be integrated out. These corrections can be computed explicitly, see for example [34, 35], but they will not be essential in context of this paper. As usual, the conformal rotation of the four-dimensional metric performed in eq. (2.14) is required if the action (2.15) is to have a canonically normalized Einstein term. Equivalently, one could work with a Brans-Dicke type action with the time variation of the Newton constant then controlled by the modulus T. In this paper, we prefer to consider the standard Einstein-frame action in four-dimensions.

The action (2.15) contains four-dimensional gravity plus various scalar fields, and represents our model for "standard cosmology".

Now that we have presented the five-dimensional theory and its four-dimensional effective theory in general, let us see how the relationship works on the level of individual solutions. We start with an arbitrary solution of the five-dimensional theory, specified by a certain metric $g_{\alpha\beta}$, a bulk scalar field ϕ and boundary scalars ϕ_i. Of course, most of the five-dimensional solutions cannot be understood as solutions of the four-dimensional effective theory, simply because they cannot be cast in the form (2.13). For example, if the brane stress energy is large, the five-dimensional solution will have a strong dependence on the orbifold coordinate which prevents it from being "near" the flat vacuum. This means, in particular, that cosmological backgrounds describing the evolution of the universe may (and typically will) start in a regime which is genuinely higher-dimensional, that is, which has no description in terms of the zero-mode action (2.15) or any other four-dimensional action with a finite number of fields. However, under specific conditions a five-dimensional solution may have a four-dimensional description. How can this be explicitly tested? Let us split our five-dimensional solution according to

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta}\,, \qquad (2.17)$$

$$\phi = \bar{\phi} + \tilde{\phi} \qquad (2.18)$$

into bared fields corresponding to the orbifold average of the full fields and the y-dependent remainder denoted by a tilde. Obviously, this decomposition can be applied to any $D = 5$ solution and the first, averaged part is automatically in the near-vacuum form (2.13). To have a four-dimensional description we should then obviously require that the y-dependent part is somehow small. A practical way of checking this is to insert the decomposition (2.17) and (2.18) into the five-dimensional equations of motion (2.3)–(2.11) and to require the terms involving tilded fields to be negligible compared to terms involving only bared fields. The remaining, bared terms then basically correspond to the equations of motion associated to the four-dimensional effective theory (2.15) showing that the averaged fields are an approximate solution of this effective theory. As an example, typical conditions to be checked are

$$\begin{aligned} e^{\tilde{\beta}} &\ll 1\,, \\ \dot{\tilde{\beta}} &\ll \dot{\bar{\beta}} \end{aligned} \tag{2.19}$$

and similar ones for the metric components. If these conditions are satisfies a four-dimensional description of the solution in terms of the effective theory (2.1) is possible. Clearly, for many five-dimensional solutions one expects those conditions to hold only in certain regions of the parameter and coordinate space. For example, a solution which is genuinely five-dimensional at early time may evolve to become effectively four-dimensional later on. It is precisely this transition between five and four dimensions that we are after in the present paper. More specifically, we would like to analyze the possibility of such a transition in the context of inflationary solutions to our five-dimensional theory (2.1).

3. Five-dimensional solutions and their four-dimensional limits

In this section we will examine cosmological solutions to our action (2.1) with all scalar fields and, hence, the potentials V, V_i being constants. This is the first step towards finding inflationary backgrounds. Varying potentials through scalar field slow-roll will be implemented as a second step later on. Such cosmological solutions to the action (2.1) coupling to constant potentials have been first presented in ref. [7] for the case of vanishing bulk potential. Subsequently, other solutions with $(A)dS_5$ and $(A)dS_5$-Schwarzschild bulk geometry have been found, see, for example refs. [8, 9, 10, 11, 15, 16]. Here, we focus on the class of solutions with $(A)dS_5$ bulk geometry and we will systematically list all possibilities. Particular emphasis is put on examining properties which are of relevance for our subsequent discussion such as existence of static-orbifold solutions and allowed ranges for the potentials V and V_i. Following our general prescription outlined in the previous section, we will also check the existence of four-dimensional limits and calculate the corresponding four-dimensional solutions where appropriate.

3.1 General properties

In order to solve the equations of motion, it is essential to note the existence of a first integral [36] which, in conformal gauge $\nu = \beta$, is given by

$$\dot{\alpha}^2 - \alpha'^2 = \left(\frac{V}{12} + 4Ce^{-4\alpha}\right) e^{2\beta}\,. \tag{3.1}$$

Here, C is an arbitrary integration constant and solutions with $(A)dS_5$ bulk geometry can be singled out by setting $C = 0$. The details of the classification are somewhat

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Bulk Potential	*Brane Potentials*	*Constraints?*
$V > 0$	$V_1 + V_2 > 0$	no non-singular solutions
$V > 0$	$V_1 + V_2 < 0$	no further constraints
$V < 0$	$V_1 + V_2 > 0$	no non-singular solutions known
$V < 0$	$V_1 + V_2 < 0$	$\lvert V_i\rvert > \sqrt{12\lvert V\rvert}$; $V_1V_2 < 0$

Table 1: Solutions with a static orbifold radius.

technical and are, therefore, provided in the appendices A, B and C. Here, we would like to summarize the main results. The structure of the solution suggests four sub-classes specified by the signs of V and $V_1 + V_2$. These sub-classes have quite different properties, particularly with respect to the existence of static-orbifold solutions. The situation concerning such solutions is summarized in table 1. It is particularly remarkable that no non-singular static-orbifold solution exists in the first case, where $V > 0$ and $V_1 + V_2 > 0$. Hence in case of all potentials being positive, a situation which seems to be quite attractive in view of inflation, the orbifold is necessarily evolving in time. It is important to realize that these results originate from global properties of the embeddings of the branes in the background $(A)dS_5$ geometry. In particular, they would be missed if we were to restrict our attention to just one of the branes. We also demonstrate in appendix C that singular solutions with static orbifold and $V > 0$, $V_1 + V_2 > 0$ exist. The singularity then typically implies a horizon separating the two branes. However, we will not make any explicit use of such exotic solutions in the following. A similar no-go theorem excludes the existence of static-orbifold solutions in the case $V < 0$, $V_1 + V_2 > 0$ as long as the metric is restricted to conformal gauge. However, the obstruction disappears when arbitrary gauge choices are considered and, although we have not constructed an explicit example, we expect static-orbifold solutions to exist under those more general circumstances.

The situation concerning arbitrary solutions with static or non-static orbifold is presented in table 2. We note that there are no further constraints on the potential values as long as $V > 0$ so that solutions exist for all values $V > 0$ and V_i arbitrary. Particularly, we have found solutions for $V > 0$, $V_1+V_2 > 0$ which are non-separating and have a dynamical orbifold in accordance with the above no-go theorem. One notes that the situation is quite different for $V < 0$ where the potentials are further constrained by $|\rho_i| \ll 1$ (where $\rho_i \equiv V_i/\sqrt{12|V|}$) and $V_1V_2 < 0$.

Which of these solutions are of interest for our subsequent discussion? As we have mentioned, we would like to consider solutions which have a chance of evolving "near" the flat vacuum state of our five-dimensional theory. A necessary condition for this to happen is that a given solution becomes effectively four-dimensional in a particular region of the parameter space spanned by V, V_i. In this section, this will be checked following the procedure described in section 2.2. Whether an existing

Bulk Potential	*Brane Potentials*	*Constraints?*
$V > 0$	$V_1 + V_2 > 0$	no further constraints
$V > 0$	$V_1 + V_2 < 0$	no further constraints
$V < 0$	$V_1 + V_2 > 0$	$\lvert V_i\rvert > \sqrt{12\lvert V\rvert}$; $V_1V_2 < 0$
$V < 0$	$V_1 + V_2 < 0$	$\lvert V_i\rvert > \sqrt{12\lvert V\rvert}$; $V_1V_2 < 0$

Table 2: General Solutions.

$D = 4$ limit, obtained if potentials are adjusted to a specific range "by hand", can actually be reached dynamically is a different matter and will be analyzed later in the context of slow-roll evolution.

It seems unlikely, that solutions with $V < 0$ will evolve towards a flat $D = 4$ limit and there is, in fact, a formal argument supporting this. As is intuitively clear and will be exemplified below, that one typically needs to be in the limit $|\rho_i| \ll 1$ of the parameter space to have a $D = 4$ description for a particular solution. However, as we have seen, solutions for $V < 0$ exist only provided that $|\rho_i| \geq 1$. We, therefore, remain with the solutions for $V > 0$ which do exist in the limit $|\rho_i| \ll 1$. During slow-roll, one expects that the scalar fields will roll downhill, so the criteria that we wish to end up near a state where all potentials are zero seems to indicate that we should consider the case where all potentials are positive initially. These simply dynamical considerations seems to restrict us further to the case where $V > 0$ and $V_1 + V_2 > 0$. In fact we shall see that there is also another, somewhat surprising possibility, namely that a negative brane potential may be pushed 'uphill' towards zero. This then allows consideration of a solution where $V > 0$ but $V_1 + V_2 < 0$. Let us, therefore, discuss the two cases $V > 0$, $V_1 + V_2 > 0$ and $V > 0$, $V_1 + V_2 < 0$ separately in some detail.

3.2 Solution with $V > 0$ and $V_1 + V_2 > 0$

The solutions of this type are explicitly given in appendix B and repeated here for convenience

$$ds_5^2 = \frac{1}{f^2}\left\{\frac{1}{k^2}\xi^2 e^{2\xi\tau}(-d\tau^2 + dy^2) + d\mathbf{x}^2\right\}, \tag{3.2}$$

where

$$f = \sigma + e^{\xi\tau}\cosh\left[(A_1 + A_2)\frac{y}{R} - A_1\right], \tag{3.3}$$

$$\xi = \frac{|A_1 + A_2|}{R}, \tag{3.4}$$

$$A_i = \operatorname{arcsinh}(\rho_i) \tag{3.5}$$

and $k = \sqrt{V/12}$. If we want $V_1 + V_2 > 0$, then f must be negative in order for the solution to fit to the boundary conditions, as shown in appendix B. This excludes the choice $\sigma = 0$, which would lead to a separating solution with static orbifold. Instead the constant σ must be strictly negative and we have a non-separating solution with a dynamical orbifold. Although the metric (3.2) looks complicated, it is just dS_5 as may be explicitly verified using the coordinate transformations given in appendix C. We note, that the solution as given in appendix B leaves the sign of ξ ambiguous. Here, we have focused on the case $\xi > 0$ since it corresponds to an expanding three-dimensional universe. The opposite case, $\xi < 0$, simply represent the time-reversal of this solution.

For any choice of (negative) σ, the metric (3.2) represents a valid solution whilst f remains negative, that is, for time values τ satisfying

$$|\xi|\tau \in \left(-\infty, \ln\left[\frac{-\sigma}{\sqrt{1+\rho_{\max}^2}}\right]\right) \tag{3.6}$$

and it becomes singular towards each of these limits. Calculating the expansion rates of the brane and orbifold scale factors we find

$$e^{-\nu}\dot{\alpha}\,|_{y=y_i} = k\sqrt{1+\rho_i^2}\,, \tag{3.7}$$

$$e^{-\nu}\dot{\beta} = -k\sigma e^{-k\tau} \tag{3.8}$$

and therefore the orbifold is expanding more rapidly than the branes throughout the range of τ for which this solution exists.

There is an elegant geometric interpretation for solutions of $(A)dS_5$ type. For the case $V < 0$ this was first discussed in ref. [15] and one can proceed in an analogous way if $V > 0$. For our example, the branes are dS_4 hypersurfaces embedded in a background dS_5, with the bulk being that portion of dS_5 which lies between the two branes. The model may be visualized as two hyperbolae of curvature $k\sqrt{1+\rho_i^2}$ drawn on the surface of a hyperboloid of curvature k embedded in a six-dimensional Minkowski space. The initial singularity at $\tau = -\infty$ corresponds to an intersection of these two hyperbolae, from which they then diverge. The details of this embedding into six-dimensional Minkowski space are given in appendix C.

Let us now explicitly carry out the metric-splitting procedure to check for possible $D = 4$ limits, as described in section 2. We write

$$\begin{aligned}\alpha &= -\ln\left[-g\right] - \ln\left[\frac{f}{g}\right] \\ &= \bar{\alpha} + \tilde{\alpha}\,,\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\beta &= -\ln\left[-\frac{k}{\xi}ge^{-\xi\tau}\right] - \ln\left[\frac{f}{g}\right] \\ &= \bar{\beta} + \tilde{\beta}\,,\end{aligned} \tag{3.10}$$

where g is defined as[2]

$$g = \sigma + e^{\xi\tau}\,. \tag{3.11}$$

In order for the y-independent terms $\bar{\alpha}$ and $\bar{\beta}$ to be a good approximation to the true solution, the equations of motion show that we should require $\dot{\tilde{\alpha}} \ll \dot{\bar{\alpha}}$, $\ddot{\tilde{\alpha}} \ll \ddot{\bar{\alpha}}$, $\dot{\tilde{\beta}} \ll \dot{\bar{\beta}}$, $\ddot{\tilde{\beta}} \ll \ddot{\bar{\beta}}$ and $e^{\tilde{\beta}} \sim 1$. These conditions show that there is an upper critical time limit prior to which this description must become valid, given by

$$\tau \ll \tau_{\rm c} \equiv \frac{1}{\xi}\ln\left[\sigma - \frac{2\sigma}{\sqrt{1+\rho_{\rm max}^2}}\right], \tag{3.12}$$

where $\rho_{\rm max}$ is the larger of the two ratios of the potentials. This expression only exists if $\rho_{\rm max} < \sqrt{3}$, so there can only ever be a four-dimensional description if both ratios $|\rho_i|$ are less than $\sqrt{3}$. If we want this description to be valid at late times, then we will in fact need $\rho_i \ll 1$.

The interpretation is simple. When $\tau \ll \tau_{\rm c}$ the solution admits a four-dimensional description. This is because the exponential in the denominator of the true solution (3.2) is extremely small when $\tau \sim -\infty$ and hence the metric may be taken as y-independent. As τ progresses from $-\infty$, this exponential becomes larger making the y-dependent cosh term more crucial, until, as $\tau \to \tau_{\rm c}$, a four-dimensional description is no longer possible. This can also be understood in terms of the geometrical interpretation. At early times, shortly after the brane-collision, the bulk samples only a small portion of the dS_5 hyperboloid and, therefore, appears approximately flat in the orbifold direction. As the branes diverge this portion increases and, after the critical time, becomes so large that the $D = 4$ approximation breaks down.

It will prove useful later to interpret (3.12) in terms of co-moving time on one of the branes. The induced metric on the branes takes the form

$$ds^2_{{\rm brane},i} = -dt^2 + \exp\left(2kt\sqrt{1+\rho_i^2}\right)d\mathbf{x}^2 \tag{3.13}$$

with $t \in (-\ln(-\sigma), \infty)$ between the initial and final singularities. In terms of co-moving time on this brane, a four-dimensional description of the complete theory is valid until t approaches a critical time

$$t \longrightarrow t_{\rm c} = \frac{1}{k\sqrt{1+\rho_i^2}}\ln\left[\frac{1}{\sigma - \sigma\sqrt{1+\rho_i^2}}\right] \tag{3.14}$$

which again only falls in the range of t covered by our solution if $\rho_i < \sqrt{3}$.

[2]There is a subtlety here. Why have we chosen to represent the cosh by 1 in g? Actually, it is possible to replace the cosh by any number c with $1 \leq c \leq \sqrt{1+\rho_{\rm max}^2}$. Such an alternative replacement would change the details of the following calculation; in particular there is some freedom in the value of the critical time $\tau_{\rm c}$. However, it will not affect our main results.

If the above constraints are satisfied, then eq. (2.14) shows that this solution has an effective four-dimensional description with metric

$$ds_4^2 = -\frac{\xi}{k}\frac{1}{(\sigma e^{-\xi\tau}+1)^3}\left(-\frac{\xi^2}{k^2}d\tau^2 + e^{-2\xi\tau}d\mathbf{x}^2\right) \tag{3.15}$$

and T-modulus

$$T = -\frac{\xi}{k}\frac{1}{(\sigma e^{-\xi\tau}+1)}\,. \tag{3.16}$$

The conformal factor makes it difficult to express the metric in terms of four-dimensional co-moving time t_4 for all τ. It is clear, though, that is does not represent dS_4 but is rather conformal to dS_4 due to our Weyl rotation to the $D = 4$ Einstein frame. At any rate, the above metric and T modulus represent an exact solution of the $D = 4$ effective action (with the other scalar fields being constant), as can be explicitly checked. Further, it is easy to verify that as $\tau \to 0$ (a limit we may only consider because the requirement $\rho_i \ll 1$ means that the original metric does not become singular until $\tau \to 0$) the four-dimensional metric (3.15) displays power-law expansion with scale-factor $\sim t_4^3$ in terms of the four-dimensional co-moving time t_4.

In summary, we have seen the first example of a dS_5 brane-world solution which displays a four-dimensional limit under certain conditions. As expected we found that the parameters $|\rho_i|$ have to be sufficently small for such a four-dimensional interpretation. In addition, however, this must arise prior tp a certain critical time beyond which no 4D limit exists. This additional time constraint is somewhat troubling given that one would like the late universe to be effectively four-dimensional. Unfortunately, it originates from the rapid expansion of the orbifold which cannot be avoided as long as $V > 0$ and $V_1 + V_2 > 0$.

3.3 Solutions with $V > 0$, $V_1 + V_2 < 0$

As mentioned earlier, naïvely one expects that in order to slow-roll down towards the Minkowski vacuum, it is necessary to consider the previous solution where all the potentials were positive. However, surprisingly it is also possible to approach the Minkowski vacuum from a state where one or both brane potentials are negative. We will discuss exactly how this may occur later, but first let us present the particular background metrics that we wish to consider and discuss their four-dimensional limits.

The metric here is closely related both to the previous solution (3.2) and to the solutions for $V < 0$ presented in ref. [9]. Again, focusing on the cases with expanding three-dimensional universe, we choose the solutions with $\xi > 0$ given in appendix B. Explicitly, they read with

$$ds_5^2 = \frac{1}{f^2}\left\{\frac{1}{k^2}\xi^2 e^{2\xi\tau}(-d\tau^2 + dy^2) + d\mathbf{x}^2\right\} \tag{3.17}$$

with f, ξ, A_i and k as defined in the previous subsection. For $V_1 + V_2 < 0$, the function f must now be positive as can be seen from appendix B. Hence, unlike for the case $V_1 + V_2 > 0$, this does not imply any restrictions on the constant σ, and for any choice of σ the metric (3.17) will be valid throughout the range of τ that is compatible with the requirement $f > 0$. If $\sigma \geq 0$, then it is valid for all $\tau \in (-\infty, \infty)$. If $\sigma < 0$, there is a singularity which leads to a semi-infinite time range

$$\xi\tau \in \left(-\infty, \ln\left[\frac{\sqrt{1+\rho_{\rm min}^2}}{|\sigma|}\right]\right) . \tag{3.18}$$

Note in particular, that we are now free to choose $\sigma = 0$ which leads to a separating solution with static orbifold. A co-ordinate transformation of the form given in appendix C shows that this metric also has bulk geometry dS_5.

The brane and orbifold Hubble rates are found to be

$$e^{-\nu}\dot{\alpha}\big|_{y=y_i} = k\sqrt{1+\rho_i^2}\,, \tag{3.19}$$

$$e^{-\nu}\dot{\beta} = -\sigma k e^{k\tau}\,. \tag{3.20}$$

The three-dimensional space is always expanding, while the orbifold radius expands, contracts or is stationary depending on whether σ is negative, positive or zero, respectively. In the stationary case $\sigma = 0$, the physical orbifold radius is easily shown to be

$$R_{\rm phys} = \left|\frac{2}{k}\sum_{i=1}^{2}\arctan\tanh\left(\frac{A_i}{2}\right)\right| . \tag{3.21}$$

Again, we can give a geometrical description along the lines of ref. [15]. Realising dS_5 as a hyperboloid in six-dimensional Minkowski space, the dS_4 branes can be obtained as intersections of certain hyperplanes with this hyperboloid. If those two hyperplanes, or, equivalently, their normal vectors, are parallel we have a static orbifold solution which corresponds to setting $\sigma = 0$. Conversely, non-parallel normal vectors indicate a dynamical orbifold and correspond to solutions with $\sigma \neq 0$. The mathematical details of this interpretation are worked out in appendix C.

As an aside, we note that for $\sigma = 0$ it is straightforward to generalize the metric (3.17) to include three-dimensional spatial curvature. One merely replaces $e^{\xi\tau}$ by $\cosh(\xi\tau)$ or $\sinh(\xi\tau)$ for positive and negative spatial curvature, respectively. By doing so it is clear that as τ progresses, three-dimensional spatial curvature becomes negligible in this solution and working with flat three-dimensional sections, as we do, is a good approximation.

Let us again explicitly follow the metric splitting procedure outlined in section 2. This gives

$$\begin{aligned}\alpha &= -\ln[g] - \ln\left[\frac{f}{g}\right]\\ &= \bar{\alpha} + \tilde{\alpha}\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}\beta &= -\ln\left[\frac{k}{\xi}\, g e^{\xi\tau}\right] - \ln\left[\frac{f}{g}\right] \\ &= \bar{\beta} + \tilde{\beta}\,, \end{aligned} \tag{3.23}$$

where now g is defined as[3]

$$g = \sigma + e^{\xi\tau}\,. \tag{3.24}$$

The properties of the four-dimensional limits of these solutions depend critically on σ. We will discuss the three cases separately.

- $\sigma = 0$: This is the separating case with static orbifold. The only requirement to be in a $D = 4$ limit is simply that $|\rho_i| \ll 1$. Due to separability, the higher Kaluza-Klein modes in the solution, corresponding to the y-dependent pieces, do not evolve relative to the zero modes. Hence, unlike in our previous case, there is no time limit by when the four-dimensional theory must become valid. Since the orbifold radius is static, the four-dimensional metric is just dS_4, that is,

$$\begin{aligned} ds_4^2 &= \frac{\xi}{k}\left\{-\frac{\xi^2}{k^2}d\tau^2 + e^{2\xi\tau}d\mathbf{x}^2\right\} \\ &= T\left\{-dt_4^2 + e^{2kt_4}d\mathbf{x}^2\right\} \end{aligned} \tag{3.25}$$

  and the T-modulus is fixed as a constant

$$T = \frac{\xi}{k}\,. \tag{3.26}$$

  This represents an exact solution of the $D = 4$ effective action (2.15). Hence, we have seen our first example for a solution which is genuinely five-dimensional as long as $\rho_i \geq 1$ and has an effective four-dimensional description for $\rho_i \ll 1$ without any further constraint on time.

- $\sigma > 0$: Again there is no time limit on the validity of a four-dimensional description, the only requirement being $|\rho_i| \ll 1$. The four-dimensional metric is now

$$ds_4^2 = \frac{\xi}{k}\frac{1}{(1+\sigma e^{\xi\tau})^3}\left\{-\frac{\xi^2}{k^2}d\tau^2 + e^{2\xi\tau}d\mathbf{x}^2\right\}\,, \tag{3.27}$$

  with T-modulus

$$T = \frac{\xi}{k}\frac{1}{(1+\sigma e^{\xi\tau})}\,. \tag{3.28}$$

  As before, this constitutes a solution to the $D = 4$ effective action (2.15). Again, it is difficult to write the metric (3.27) in terms of four-dimensional co-moving time t_4, but once more it is conformal to dS_4 with the T-modulus as a

[3] Again, we could choose a different constant to replace the cosh.

conformal factor. At early times when $\sigma e^{\xi\tau} \ll 1$, T is approximately constant, so the metric is close to dS_4. As $\tau \to \infty$ it is clear that $t_4 \sim -e^{-3|\xi|\tau/2}$ so that $t_4 \in (-\infty, 0)$. When approaching the upper time limit, $t_4 \to 0$, the scale-factor and the T modulus contract according to the power-laws $a_4 \sim (-t_4)^{1/3}$ and $T \sim (-t_4)^{2/3}$. Altogether, this implies that the 4D limit corresponds to a negative time-branch solution which ends in a four-dimensional curvature singularity combined with a brane collision.

- $\sigma < 0$: Here, the metric (3.17) possesses a singularity which arises at different finite times τ across the orbifold. Hence the Kaluza-Klein tower may be expected to have a significant effect on the validity of a four-dimensional description near this singularity. As for the solution with all positive potentials, this leads to a time-limit on the validity of a four-dimensional description given by

$$\tau \ll \tau_{\rm c} \equiv \frac{1}{\xi} \ln \left[\frac{1}{|\sigma|}\right] , \tag{3.29}$$

in addition to requiring that $|\rho_i| \ll 1$. In terms of co-moving time on the branes, which runs from $-\infty$ to $+\infty$, this four-dimensional description is valid until

$$t \longrightarrow t_{\rm c} = \frac{1}{k\sqrt{1+\rho_i^2}} \ln \left[\frac{1}{|\sigma|(\sqrt{1+\rho_i^2}-1)}\right] . \tag{3.30}$$

If the above conditions are satisfied, then the effective four-dimensional metric is the same as the previous case, but with $\sigma < 0$:

$$ds_4^2 = \frac{\xi}{k}\frac{1}{(1-|\sigma|e^{\xi\tau})^3}\left\{-\frac{\xi^2}{k^2}d\tau^2 + e^{2\xi\tau}d\mathbf{x}^2\right\} \tag{3.31}$$

and the T-modulus is

$$T = \frac{\xi}{k}\frac{1}{(1-|\sigma|e^{\xi\tau})} . \tag{3.32}$$

Again this solves the 4D effective action (2.15). As with the previous case, the metric behaves like dS_4 when $|\sigma|e^{|\xi|\tau} \ll 1$, and displays power-law expansion at later times. Here, though, four-dimensional co-moving time is in the range $t_4 \in (-\infty, \infty)$ and there is no curvature singularity at finite co-moving time. Another difference to the previous case is, of course, that the orbifold is now expanding and the four-dimensional description breaks down as τ approaches the critical time τ_c.

It may help to clarify the situation by discussing the behaviour of these solutions from the viewpoint of the resulting four-dimensional action (2.15) and the associated effective potential

$$\frac{V_4}{M_5^3} = \frac{1}{2T^2}\sum_{i=1}^{2} V_i + \frac{1}{T}RV \tag{3.33}$$

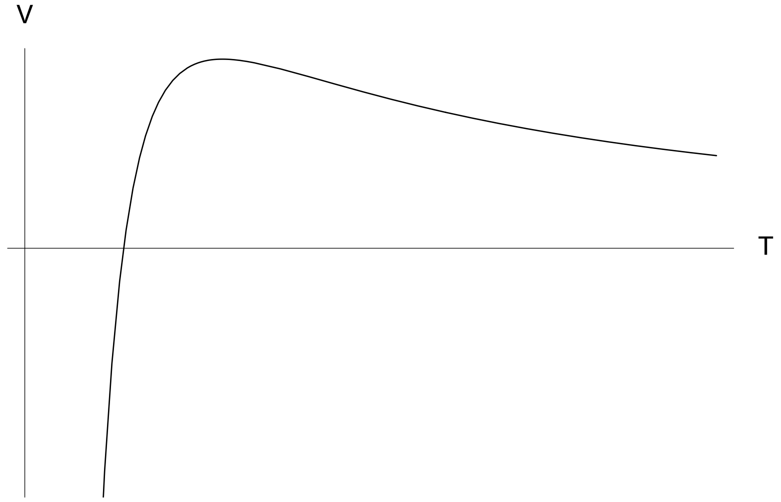

Figure 1: Shape of the effective $D = 4$ potential for the T-modulus in the 4D theory resulting from (3.17).

for the T modulus. The form of this potential, for $V > 0$ and $V_1 + V_2 < 0$ is schematically shown in figure 1. The exact shape of the potential depends, of course, on the precise values of V and $V_1 + V_2$ but, due to our choices for the signs of these quantities, it always has a maximum at

$$T = \frac{V_1 + V_2}{RV} \simeq \frac{\xi}{k}\,. \tag{3.34}$$

The various choices of σ described above can now be understood in terms of how the T-modulus is rolling in this effective potential. Let us again discuss the three cases separately.

- $\sigma = 0$: In this separating case, we have obtained an effective $D = 4$ solution with a constant T-modulus. This can only be compatible with the effective potential (3.33) if T is sitting at the maximum and a comparison of eq. (3.34) and eq. (3.26) shows that this is indeed the case. This situation is clearly unstable.[4] We stress that this instability, although quite apparent in the four-dimensional limit was not at all obvious from a purely five-dimensional viewpoint. Of course, we cannot really tell from this $D = 4$ analysis whether the solution is unstable for any value of ρ_i or whether the instability only arises in the $D = 4$ limit when $\rho_i \ll 1$. This can only be clarified by a full five-dimensional stability analysis which is beyond the scope of the present paper. In any case, our observation

[4]It is not possible to simply reverse the signs of all potentials, thus giving the T-modulus potential a minimum. As discussed, in any solution with $V < 0$ and $V_1 + V_2 > 0$ one simply never reaches the four-dimensional regime where a description in terms of $V_4(T)$ would be appropriate.

shows that classical stability is an important issue for cosmological brane-world solutions. To make this solutions realistic, clearly some additional stabilizing potential, possibly of non-perturbative origin, has to be invoked. This will be further discussed later on when slow-roll of scalar fields is analyzed.

- $\sigma > 0$: If, either through initial conditions or a fluctuation, the T-modulus were to begin to move towards zero, then we enter the class of solutions with $\sigma > 0$. As the T-modulus increases its kinetic energy, we enter a power-law expansion as is usual for exponential potentials ($T = e^{\bar{\beta}}$ and $\bar{\beta}$ has the canonically normalized kinetic term). The branes will ultimately collide at $\tau = \infty$ which corresponds to $t_4 = 0$ in terms of four-dimensional co-moving time. Realistically, one expects that once the orbifold radius is smaller than the scale of an underlying Calabi-Yau space higher-order corrections become important, which invalidate our simple lowest-order effective actions. One may hope that these corrections induce a "graceful exit" from the negative time branch, similar to what is normally assumed in pre-Big Bang cosmology [37, 38].

- $\sigma < 0$: The solutions with $\sigma < 0$ correspond to the T-modulus rolling off its maximum in the opposite direction; $T \to \infty$. Because $V(T)$ is flatter in this direction, T rolls more slowly, and so co-moving time is now valid for $t \in (-\infty, \infty)$. However, the four-dimensional description will break down as $\tau \to \tau_c$ where the higher modes in eq. (3.17) become more crucial.

4. The slow-roll approximation

In the previous section, we identified classes of solutions to the 5D Einstein equations which possessed a four-dimensional description around the Minkowski vacuum for some region of their parameter space. However, these solutions were always treated adiabatically in the sense that the brane and bulk scalar potentials were varied "by hand". Our next task is to see whether such background solutions do in fact evolve dynamically towards the four-dimensional regime. As is usual for inflation, we do not want the evolution of the scalar fields to appreciably disturb the adiabatic solutions, and so we must impose slow-roll conditions on the energy-momentum tensors. In this section we will examine the general slow-roll conditions that should be imposed on bulk and brane scalar fields for the validity of arbitrary conformal gauge $(A)dS_5$ background solutions with dS_4 branes. This includes all of the solutions of the previous section. Slow-roll conditions for a brane scalar field have already been studied in ref. [19]. However, apart from considering a pure bulk cosmological constant, ref. [19] looked at the most general conditions under which accelerated expansion of the brane scale-factors could occur. Our purpose here is somewhat different. We ultimately wish to know how the explicit solutions of the previous section evolve during slow-roll. Therefore we must be sure our brane slow-roll conditions are appropriate for dS_4 branes, and not e.g. brane power-law expansion.

For the moment, we will not restrict ourselves to any particular form for $V(\phi)$ or $V_i(\phi_i, \phi)$, although sometimes we may consider a special case where they are related. In general, we will see that it is impossible to completely separate the bulk and brane conditions. The resulting slow-roll conditions will be applied to our specific background solutions in section 5.

4.1 General requirements of slow-roll

For the bulk energy-momemtum tensor to be dominated by the scalar field potential, the Einstein eqs. (2.3)–(2.6) show that one should require

$$\left| e^{-2\nu}\dot{\phi}^2 + e^{-2\beta}\phi'^2 \right| \ll 2|V| \,, \tag{4.1}$$

$$\left| e^{-2\nu}\dot{\phi}^2 - e^{-2\beta}\phi'^2 \right| \ll 2|V| \tag{4.2}$$

and in addition

$$\left| \dot{\phi}\phi' \right| \ll 6\, |\dot{\alpha}\alpha'| \,. \tag{4.3}$$

The bulk scalar ϕ has the (conformal gauge) field eq. (2.7):

$$e^{-2\beta}(\ddot{\phi} + 3\dot{\alpha}\dot{\phi}) - e^{-2\beta}(\phi'' + 3\alpha'\phi') = -\frac{\partial V}{\partial \phi} \,. \tag{4.4}$$

Also, ϕ is subject to boundary conditions at the branes. Since these boundary conditions would enter the field equation as terms involving a δ-function, it is not possible to neglect them during slow-roll. Hence we must ensure

$$e^{-\beta}\phi' \,|_{y=y_i} = \pm\frac{1}{2}\left[\frac{\partial V_i}{\partial \phi}\right]_{y=y_i} \,. \tag{4.5}$$

The branes complicate the bulk slow-roll analysis in two ways. Firstly, for non-zero brane potentials, we cannot take ϕ to be y-independent, because there would then be nothing to compensate for the y-dependence of the metric components in eq. (4.4). Hence, because of the branes, even the slow-roll equation for ϕ must contain derivatives with respect to both y and τ. Secondly, suppose we were simply to neglect all second derivative terms in (4.4) by analogy with the standard case. It is then possible to find the most general solution to the resulting first order partial differential equation. The only way this solution can match the boundary conditions (4.5) is if $V_i(\phi, \phi_i) = f(\phi_i)[V(\phi)]^{2/3}$. If the brane and boundary potentials are related in this way, then the explicit solution for ϕ shows that it was not consistent to neglect $(\ddot{\phi} - \phi'')$ with respect to $3(\dot{\alpha}\dot{\phi} - \alpha'\phi')$. We conclude that it is not possible to neglect the second derivatives of the bulk scalar during slow-roll, and again attribute this effect directly to the presence of the branes.

The energy-momentum tensors of the brane scalar fields ϕ_i only appear in the boundary conditions on the Einstein eqs. (2.9) and (2.10). These show that for dS_4 branes one should require

$$5e^{-2v}\dot{\phi}_i^2 \ll 2V_i\,. \tag{4.6}$$

The brane scalar fields ϕ_i each have field eq. (2.8):

$$e^{-2\nu}\left[\ddot{\phi}_i + (3\dot{\alpha} - \dot{\nu})\dot{\phi}_i\right] = -\frac{\partial V_i}{\partial \phi_i}\,, \tag{4.7}$$

where all the components of the metric are evaluated at the brane positions $y = 0, R$. This equation has only one independent variable, τ, which makes finding the appropriate slow-roll conditions very much analogous to the standard case, and certainly much simpler than the bulk scalar. Let us therefore begin by examining the brane conditions.

We split the left hand side of (4.7) in a gauge-invariant way and neglect $\left|e^{-\nu}\partial_\tau(e^{-\nu}\dot{\phi}_i)\right|$ to find the slow-roll equation

$$3e^{-2\nu}\dot{\alpha}\,\dot{\phi}_i = -\frac{\partial V_i}{\partial \phi_i}\,. \tag{4.8}$$

Exactly as expected, (4.8) shows that if $e^{-\nu}\dot{\alpha} > 0$, then during slow-roll the brane scalar fields also roll downhill, irrespective of the sign of their potentials.

If T_{05} is negligible, which we will need to check using the bulk slow-roll conditions, then we may use the first integral (3.1). For $C = 0$ and general time gauge, it be written as

$$e^{-2\nu}(\dot{\alpha}^2 - \alpha'^2) = \frac{V}{12}\,. \tag{4.9}$$

We see that the brane Hubble rates are $e^{-\nu}\dot{\alpha} = \sqrt{\frac{V}{12} + \frac{V_i^1}{144}}$ for any $(A)dS_5$ background. To apply this to slow-roll, we assume that to zeroth order the metric components in eq. (4.8) may be taken to have their background values as if the potentials were constant. Equations (4.6) and (4.8) then give the brane ϵ-condition

$$\epsilon_i \equiv \frac{10}{3V_i}\frac{1}{\left(V + \frac{V_i^2}{12}\right)}\left(\frac{\partial V_i}{\partial \phi_i}\right)^2 \ll 1\,. \tag{4.10}$$

This ϵ-condition is appropriate for all of the solutions of section 3.

As usual, consistency of neglecting the second derivative terms in (4.7) generates the η-condition. Consistency of neglecting these terms gives

$$\left|\frac{4}{3\left(V + \frac{V_i^2}{12}\right)}\frac{\partial V_i}{\partial \phi_i}\frac{\partial^2 V_i}{\partial \phi_i^2} + \frac{e^{-\nu}\dot{\phi}}{3\sqrt{\frac{V}{12} + \frac{V_i^2}{144}}}\frac{\partial^2 V_i}{\partial \phi\,\partial \phi_i}\right| \ll \left|\frac{\partial V_i}{\partial \phi_i}\right|, \tag{4.11}$$

where we have again taken the zeroth order approximation of treating the background metric as constant. The first term on the l.h.s. of eq. (4.11) clearly forms the brane η-condition and is valid for all dS_4 branes in any background $(A)dS_5$ metric:

$$\eta_i \equiv \left| \frac{4}{3\left(V+\frac{V_i^2}{12}\right)} \frac{\partial^2 V_i}{\partial \phi_i^2} \right| \ll 1 \,. \tag{4.12}$$

The second term in eq. (4.11) takes account of the fact that the brane potential also depends on ϕ and must be related to the bulk slow-roll conditions. Hence we see that in general, slow-roll on the brane and in the bulk is inextricably linked.

Notice that the brane slow-roll conditions break down when the potentials approach either $V = V_i = 0$ or $V = -V_i^2/12$. This confirms that slow-roll of the ϕ_i's will end when the potentials approach either the Minkowski or Randall-Sundrum values. In addition, as noticed in refs. [19, 20], since ϵ_i and η_i are more strongly suppressed in the five-dimensional regime than in the standard case, it is possible for certain types of potential that are unable to support standard four-dimensional inflation to nonetheless satisfy (4.10). Slow-roll of the brane scalars could then end naturally if such a model evolved towards a four-dimensional regime, where the denominator of (4.10) becomes small, without having to require any special feature in the potential. We will not investigate this possibility further here.

Let us now return to the bulk scalar field. As noticed above, it is not possible to take ϕ to be y-independent, nor to consistently neglect the second derivatives in its field equation. To be able to handle these complications, we introduce a function $\mathcal{T}$ by $\phi = \phi(\mathcal{T}(\tau, y))$ where

$$\frac{d\phi}{d\mathcal{T}} \equiv -\frac{3}{V}\frac{\partial V}{\partial \phi} \tag{4.13}$$

which fixes $\mathcal{T}$ (up to a constant) for any given potential. Notice that since ϕ is dimensionless, $\mathcal{T}$ is also. This transforms the bulk scalar slow-roll field equation to

$$\ddot{\mathcal{T}} + 3\dot{\alpha}\dot{\mathcal{T}} - \mathcal{T}'' - 3\alpha'\mathcal{T}' = e^{2\beta}\frac{V}{3} \,, \tag{4.14}$$

where in transforming $\ddot{\phi} - \phi''$ we have neglected

$$\left| \frac{d^2\phi}{d\mathcal{T}^2}\left(\dot{\mathcal{T}}^2 - \mathcal{T}'^2\right) \right| \ll \left| \frac{d\phi}{d\mathcal{T}}\left(\ddot{\mathcal{T}} - \mathcal{T}''\right) \right| \tag{4.15}$$

which will later form the η-condition. The solution of eq. (4.14) tells us how $\mathcal{T}$ behaves, which in turn would describe ϕ were we to specify a form for $V(\phi)$ and invert (4.13). It is obviously extremely difficult to solve eq. (4.14) exactly, since the metric components α and β also vary due to the slow-roll of the potentials. As a first approximation, we may treat the potentials in these terms as constant. Remarkably, it is then possible to find the particular integral of eq. (4.14). From

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eq. (A.2) in appendix A we see directly that the particular integral is $\mathcal{T} = \alpha$ for arbitrary conformal gauge backgrounds with $(A)dS_5$ bulk geometry. Clearly, this includes all of the solutions discussed in section 3

In order to allow $\mathcal{T}$ to match to arbitrary boundary conditions, one should also consider the homogeneous equation

$$\ddot{\mathcal{T}} + 3\dot{\alpha}\dot{\mathcal{T}} - \mathcal{T}'' - 3\alpha'\mathcal{T}' = 0 \tag{4.16}$$

so as to obtain the general solution to the slow-roll field equation. As shown in appendix A, in conformal gauge the metric takes the form (A.4):

$$ds^2 = \frac{1}{(f_+ + f_-)^2}\left\{-\frac{48}{V} f_+' f_-' \, dx_+ dx_- + d\mathbf{x}^2\right\} \tag{4.17}$$

with f_+ and f_- functions of $x_+ = \tau + y$ and $x_- = \tau - y$, respectively. It is possible to solve the homogeneous equation by defining new co-ordinates (canonical ones for $(A)dS$) as

$$X^+ = \sqrt{\frac{48}{|V|}} f_+ \,, \tag{4.18}$$

$$X^- = \sqrt{\frac{48}{|V|}} f_- \tag{4.19}$$

and $X^\pm = X_0 \pm X_1$. In terms of these co-ordinates, the homogeneous eq. (4.16) takes the form

$$\frac{\partial^2 \mathcal{T}}{\partial {X_0}^2} - \frac{3}{X_0}\frac{\partial \mathcal{T}}{\partial X_0} - \frac{\partial^2 \mathcal{T}}{\partial {X_1}^2} = 0 \tag{4.20}$$

for any conformal gauge $(A)dS_5$ background solution. Equation (4.20) is now solved by separation of variables and the result may be expressed in terms of a series of Bessel and Neumann functions. The general solution to eq. (4.14) is then

$$\begin{aligned}\mathcal{T}(\tau, y) = {} & \alpha(\tau, y) + \left[\lambda_0^a + \lambda_0^b X_0^4\right]\left[\lambda_0^c + \lambda_0^d X_1\right] + \\ & + \sum_{m \neq 0} X_0^2 \left[\lambda_m^a J_2(m X_0) + \lambda_m^b N_2(m X_0)\right] \times \\ & \times \left[\lambda_m^c \sin(m X_1) + \lambda_m^d \cos(m X_1)\right] \,,\end{aligned} \tag{4.21}$$

where $X_{0,1}$ are the functions of τ and y given in eqs. (4.18) and (4.19) and m is an arbitrary separation constant. In principle, the integration constants λ in this series may be used to satisfy arbitrary boundary conditions on $\mathcal{T}$. In practise, the transformations (4.18) and (4.19) deform the shape of the branes such that they do not, in general, lie along lines of constant X_0 or X_1. For arbitrary background metrics, this can make it somewhat complicated to fit the general solution (4.21) to the boundary conditions. Two possible simplifications immediately present themselves; either we can consider arbitrary background solutions with a specific relation between the brane and bulk potentials, or we can keep all the potentials arbitrary but restrict the class of background solution. Let us consider each of these cases in turn.

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4.2 Special solution for related bulk and brane potentials

In this case we simply take the trivial solution to the homogeneous eq. (4.16) and hence have $\mathcal{T} = \alpha$. Obviously this will not be possible for arbitrary $V_i(\phi, \phi_i)$ and indeed the boundary conditions (4.5) and (2.9) show that we must choose the brane and bulk scalar potentials to be related as

$$V_i(\phi, \phi_i) = f(\phi_i)\,[V(\phi)]^{1/2}\,. \tag{4.22}$$

However, we do not need to make any new assumptions about our background. What follows is true for any $(A)dS_5$ bulk in conformal gauge.

With $\mathcal{T} = \alpha$, the slow-roll condition (4.1) forms the bulk ϵ-condition

$$\epsilon \equiv \frac{3}{8V^2}(1+2\rho_{\rm max}^2)\left(\frac{\partial V}{\partial \phi}\right)^2 \ll 1 \tag{4.23}$$

and the other two conditions (4.2) and (4.3) are always less restrictive than this. Equation (4.23) is closely parallel to the usual ϵ-condition, with corrections depending on the strength of the brane sources. If (4.23) is satisfied, then (4.3) shows that T_{05} is negligible, providing justification for our use of the first integral (4.9) in deriving the brane ϵ-condition.

The bulk η-condition comes from consistency of neglecting the piece of the second derivatives of ϕ in eq. (4.15). In this special case we may again use the first integral eq. (4.9). This allows us to rewrite the condition (4.15) as

$$\left|\frac{3}{V^2}\left(\frac{\partial V}{\partial \phi}\right)^2 - \frac{3}{V}\frac{\partial^2 V}{\partial \phi^2}\right| \ll 1\,. \tag{4.24}$$

The first term in this expression is very similar to the ϵ-condition. However, unlike the standard case in 4D, it is not always true that this term will be small whenever the ϵ-condition is satisfied. In fact, this first term is a more restrictive ϵ-condition whenever $\rho_i^2 < 7/2$. Bearing this in mind, we have the bulk η-condition

$$\eta \equiv \left|\frac{3}{V}\frac{\partial^2 V}{\partial \phi^2}\right| \ll 1\,. \tag{4.25}$$

Finally, the second term in eq. (4.11) becomes

$$\left|\frac{\partial_\phi V}{\partial_{\phi_i} V_i}\frac{\partial^2 V_i}{\partial \phi \partial \phi_i}\right| \ll 1\,. \tag{4.26}$$

Unlike the brane slow-roll conditions, we see that the bulk conditions only break down near the Minkowski vacuum, and not the Randall-Sundrum vacuum. This is to be expected, as the Randall-Sundrum vacuum is still AdS_5 from the bulk point of view.

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4.3 General solution for backgrounds with a static radius

For backgrounds where the orbifold radius is static, it is possible to match the general solution to arbitrary boundary conditions, and so consider cases with arbitrary potentials. Here we present the conditions that are required in this case with values of $\rho_i \sim \mathcal{O}(1)$ or smaller. We refer the interested reader to appendix D for a derivation and more general results for arbitrary ρ_i.

The brane ϵ and η conditions are again given by eqs. (4.10) and (4.12). This is to be expected as no assumptions were made about the background in deriving these conditions. The bulk field now has a slow-roll equation

$$e^{-\nu}\dot{\phi} = -4\sqrt{\frac{V}{12}}\left(\frac{1}{V}\frac{\partial V}{\partial \phi} - \frac{1}{2(V_1+V_2)}\frac{\partial}{\partial \phi}(V_1+V_2)\right). \tag{4.27}$$

Inserting this into the requirements (4.1), (4.2) and (4.3) gives the conditions

$$\epsilon \equiv \frac{2}{3V^2}\left(\frac{\partial V}{\partial \phi}\right)^2 \ll 1\,, \tag{4.28}$$

$$\epsilon_{12} \equiv \frac{1}{24(V_1+V_2)^2}\left(\frac{\partial}{\partial \phi}(V_1+V_2)\right)^2 \ll 1\,. \tag{4.29}$$

The first of these conditions ensures slow-roll of the bulk scalar in its own potential, and is closely analogous to the standard condition. The second ensures that the evolution of the bulk scalar does not cause the brane potentials to come out of slow-roll. Clearly, each of these should possess a corresponding η-condition. The η-condition for the bulk field alone is again found by consistency of eq. (4.15) and requires

$$\eta \equiv \left|\frac{3}{V}\frac{\partial^2 V}{\partial \phi^2}\right| \ll 1 \tag{4.30}$$

exactly as in eq. (4.25) for arbitrary backgrounds with related brane and bulk potentials. This is again expected. The η-condition for the bulk field alone does not notice the presence of the branes. An η-condition corresponding to condition (4.29) should also be imposed in principle. This is once more complicated, and here we simply remark that potentials which satisfy all the other conditions are also likely to satisfy this.

5. Application to the background solutions

The formalism of the previous section has introduced slow-roll parameters ϵ and η for the bulk and brane scalar fields. In addition, we have confirmed the intuition that, during dS inflation, scalar fields roll downhill towards lower values of their potentials. We now wish to apply this to the solutions of section 3, thus upgrading

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them from static background spaces to inflationary models. Following the description in section 2, we wish to look for solutions that evolve towards the Minkowski vacuum state with $V = V_i = 0$. Notice that if we were to look for a solution that evolved towards the Randall-Sundrum vacuum with $V < 0$ and $V_1 = -V_2 = \pm\sqrt{12|V|}$, then naïvely we would need to start in a region with the negative brane potential at a value $|V_-| < \sqrt{12|V|}$ so as to roll downhill towards the vacuum state. Table 2 shows that there is no such solution.

How can we decide whether a particular brane-inflationary model is a viable candidate for cosmology? This depends on when, during the evolution of the universe, we envisage that the full five-dimensional solutions entered the 4D regime. We wish to distinguish three possibilities:

- The brane sources are negligible either before inflation begins, or within a small number of e-folds. This could arise because of some unknown constraint from the underlying heterotic M-Theory, or simply because of the initial conditions in our patch of the universe. Since the model is then four-dimensional from the beginning of inflation, it is unlikely that there are any direct cosmological signatures of extra-dimensions. In order to detect the higher-dimensional theory, one would have to hope that it had somehow left an imprint on the structure of the low-energy effective action.

- The universe becomes four-dimensional during inflation, but after a reasonable number of e-folds of expansion have occurred. So long as inflation ended soon after (certainly within another ~ 65 e-folds), then a significant fraction of our present Hubble volume will have originally left the horizon during five-dimensional inflation. This lays open the possibilities of observable signatures of the fifth dimension and branes in the inflationary power spectrum at large scales. In addition, since (p)reheating would occur in the four-dimensional regime [40, 41], it is conceivable that this scenario could avoid constraints (see ref. [42]) on the production of incoherent Kaluza-Klein particles.

- The universe is five-dimensional throughout inflation. Hence the initial fluctuation spectrum was entirely produced during the 5D regime. At the time of writing, experiments only place strict constraints on the universe during nucleosynthesis (see refs. [21, 22] for a review), so it is conceivable that the universe could have remained five-dimensional for part of the radiation-dominated era. Consequently, the evolution of the initial fluctuations throughout all this time could be dramatically different from the standard case [23]–[26] and hence lead to striking signatures in the CMB [39]. Such models are then of considerable interest, though they may be tightly constrained by further investigations of (p)reheating [42] or baryogenesis [43] in brane-worlds.

We will now examine the solutions presented in section 3 to discover into which of these categories they fall. We are primarily interested in finding models in the second category, because these both have fluctuations that were formed during the five-dimensional era now visible in the CMB, and are also explicitly calculable throughout their evolution.

5.1 Evolution of the positive potential solution

In section 3, we found that this solution would admit a four-dimensional description so long as the potentials satisfied $\rho_i < \sqrt{3}$ before time

$$\tau_{\rm c} = \frac{1}{|\xi|} \ln \left[\frac{2\sigma}{\sqrt{1+\rho_{\rm max}^2}} - \sigma \right], \tag{5.1}$$

where σ was a positive constant. Let us now make a rough estimate of the amount of inflation that occurs before this time. Considering simply the e-folds of expansion of the brane scale factors, we have

$$N_e \equiv \int_{t_i}^{t_f} e^{-\nu} \dot{\alpha}\, dt \leq k \sqrt{1+\rho_i^2}\, (t_f - t_i)\,, \tag{5.2}$$

where t is co-moving time measured on one of the branes. The upper limit comes from assuming that the brane Hubble rate is constant during inflation, rather than decreasing due to slow-roll. Let us naïvely extrapolate (3.2) back to the initial singularity at $t_i = -\ln\sigma$ (where the two branes were co-incident and the full eleven-dimensional theory should really have been used) and take this as the earliest inflation could begin. Requiring that inflation ends while the solution still has a four-dimensional description would then give

$$N_e \ll \ln \left[\frac{1}{\sqrt{1+\rho_i^2} - 1} \right], \tag{5.3}$$

independent of σ. For reasonable i.e. not fine-tuned) values of ρ_i this shows that the background metric (3.2) has at most $\mathcal{O}(\text{few})$ e-folds of brane expansion before the solution looks five-dimensional. If we wish to obtain sufficient inflation to solve the horizon problem, we can be sure that the solution with $V > 0$ and $V_i > 0$ will end inflation in the five-dimensional regime. This argument has ignored the effects of slow-roll, which causes ρ_i to evolve and thus affects N_e. However, it is easy to convince oneself that this will not significantly alter our conclusions.

Hence, if they are realistic at all, solutions where all the potentials are positive must fall into the third category of cosmological models; the transition to standard cosmology is made after inflation has ended. Because of the difficulty of obtaining a complete (analytic) solution when the branes are radiation-dominated [44, 45], and possibly even greater difficulty of attempting any quasi-realistic form of reheating in these models, we do not pursue this solution further here.

5.2 Evolution of the solutions with $V_1 + V_2 < 0$

Section 3.3 presented solutions (3.17) which required $V_1 + V_2 < 0$. In section 4 we proved that, as expected, brane scalar fields slow-roll towards lower values of their potentials. Therefore, if we wish to evolve towards the Minkowski vacuum, it is surprising that we can consider the metric (3.17) because its negative brane potential appears to need to roll uphill. It is possible to consider the negative potential as a pure constant. However, since $V_1 + V_2 < 0$, this will not allow us to reach the four-dimensional regime during inflation unless we have been there from the start.

The important observation is that the boundary potentials V_i may (and in Hořava-Witten theory do) also depend on the bulk scalar field. For certain choices of $V_i(\phi, \phi_i)$, the negative brane potential may indeed roll towards zero through slow-roll of ϕ down its own (positive) potential $V(\phi)$. To see how this works, remember that the metrics (3.17) require $|\rho_i| \ll 1$ for a four-dimensional description. In order for $|\rho_i|$ to decrease during slow-roll we require

$$\frac{e^{-\nu}\dot{\rho}_i}{\rho_i} = \frac{e^{-\nu}\dot{\phi}_i}{V_i}\frac{\partial V_i}{\partial \phi_i} + \frac{e^{-\nu}\dot{\phi}}{V_i}\frac{\partial V_i}{\partial \phi} - \frac{e^{-\nu}\dot{\phi}}{2V}\frac{\partial V}{\partial \phi} < 0\,. \tag{5.4}$$

If we consider the limiting case where there is no scalar field on the negative brane,[5] just an extra contribution to the bulk scalar potential, then we find that this condition is satisfied so long as $e^{-\nu}\dot{\phi} < 0$ and V_i depends on the bulk scalar more strongly than $\sqrt{V(\phi)}$. If V_i also depends on some brane scalar field, then there is an extra positive term in (5.4) so it is necessary for the ϕ-dependence to be correspondingly stronger.

In addition we need to be sure that $e^{-\nu}\dot{\phi} < 0$ so that ϕ is still rolling downhill! For the separating solution with $\sigma = 0$, the slow-roll eq. (4.27) shows ϕ will indeed roll downhill so long as V_i depends on ϕ more *weakly* than $V^2(\phi)$. For stronger dependence than this, the brane potentials will actually also drive the bulk scalar in the "wrong" direction.

The particular choice $\sigma = 0$ with potentials related by $\sqrt{V} < V_i < V^2$ therefore forms an example of the second class of cosmological models; it undergoes an arbitrarily large number of e-folds of five-dimensional inflation, before becoming four-dimensional towards the end of inflation when it may be explicitly matched on to standard inflationary models. It then seems likely that this would lead to observable signatures in the CMB, caused by the different behaviour of primordial fluctuations in the five- and four-dimensional eras [23]–[26].

However, as noticed at the end of section 3.3, the model with a static orbifold is somewhat unstable and in general we would expect fluctuations to cause us to move into the classes with $\sigma \neq 0$. If we move towards positive σ then we have a pre-Big Bang situation, whereas standard inflation still applies if we move towards $\sigma < 0$.

[5] More realistically, this simply means that any scalar fields that were present did not possess inflationary initial conditions, and so quickly became negligible.

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These solutions could then still be an example of the second class of cosmological models so long as sufficient inflation can be achieved before the four-dimensional description breaks down as $\tau \to \tau_c$. A similar calculation to the one in section 5.1 now shows that the maximum number of e-folds of inflation on the brane before this time is

$$N_e \equiv \int_{t_i}^{t_f} e^{-\nu}\dot{\alpha}\,dt \leq \ln\left[\frac{1}{|\sigma|(\sqrt{1+\rho_i^2}-1)}\right] - kt_i\sqrt{1+\rho_i^2}\,. \tag{5.5}$$

Since brane time $t \in (-\infty, \infty)$ for this solution, there is no objection to allowing inflation to start at $t_i \to -\infty$ and thus obtaining arbitrarily many e-folds of expansion whilst remaining in a four-dimensional regime.

The above conclusions of course depend on the assumption that the solutions are still inflating while until after they have become four-dimensional. In other words, we need to be sure that the slow-roll conditions can still be satisfied during the transition to the 4D regime. In addition, we must hope that inflation ends soon after this transition has occurred; another ~ 65 e-folds of standard inflation would clearly dilute any fluctuations which were sensitive to the five-dimensional regime. In the next section we will verify that these additional criteria are met in the context of two simple models.

6. Explicit examples and their phenomenology

In this section, we will focus on cases with a transition from five to four dimensions during inflation. We will study classes of inflationary models which realize this transition explicitly and discuss some of their properties. This, for the first time, opens up the possibility of connecting genuinely higher-dimensional models of inflation with the subsequent standard evolution of the universe as described by a four-dimensional low-energy effective theory. To set the scene, let us first gather some of the relevant results which we have obtained so far.

As we have seen, background solutions with all potentials positive, or more precisely with $V > 0$ and $V_1 + V_2 > 0$, do not approach the flat vacuum state during inflation. For such solutions inflation ends in a five-dimensional regime with the transition to four dimensions postponed to a later stage. The seemingly most attractive class of solutions is therefore not suitable for our purpose.

On the other hand, we clearly need the bulk potential to be positive if we want to evolve into the flat vacuum state, as we do in this paper. What remains are the solutions for $V > 0$ and $V_1 + V_2 < 0$ described in section 3.3. From our classification in appendix B, they are given by

$$ds^2 = \frac{1}{f^2}\left\{\frac{12}{V}\xi^2 e^{2\xi\tau}(-d\tau^2 + dy^2) + d\mathbf{x}^2\right\} \tag{6.1}$$

together with the definitions

$$f = \sigma + e^{\xi\tau} \cosh\left[(A_1 + A_2)\frac{y}{R} - A_1\right], \tag{6.2}$$

and

$$A_i = \text{arcsinh}(\rho_i)\,, \qquad \xi = \frac{|A_1 + A_2|}{R}\,. \tag{6.3}$$

Here σ is an arbitrary integration constant. We have shown that the above solutions approach the flat vacuum if $|\rho_i| \ll 1$ and if, for $\sigma < 0$, the additional constraint (3.29) on the time τ is satisfied. More precisely, in this $D = 4$ limit the $D = 5$ background solutions are well approximated by certain solutions to the four-dimensional effective action (2.15). As shown in section 3.3, the metric for these $D = 4$ solutions is given by

$$ds_4^2 = \frac{\xi}{k}\frac{1}{(1+\sigma e^{\xi\tau})^3}\left\{-\frac{\xi^2}{k^2}d\tau^2 + e^{2\xi\tau}d\mathbf{x}^2\right\} \tag{6.4}$$

and the T-modulus is

$$T = \frac{\xi}{k}\frac{1}{(1+\sigma e^{\xi\tau})}\,. \tag{6.5}$$

The value of the constant σ is of particular importance. For $\sigma = 0$ the five-dimensional solution is separating and has a static orbifold, a property reflected by a constant T-modulus in the associated four-dimensional solution. Unfortunately, as we have seen, the constant value for T corresponds to the maximum of the four-dimensional effective potential (3.33). Therefore, this solution is unstable at least once it approaches the $D = 4$ limit $|\rho_i| \ll 1$ and will be driven, by small perturbations, to a solution where T rolls down the potential. This is precisely what is described by the non-separating backgrounds with $\sigma \neq 0$. As can be seen from eq. (6.4) and (6.5) their four-dimensional limits correspond approximately to dS_4 with a constant T-modulus before a certain time τ and a power-law expansion with a rolling T-modulus after this time. For a realistic model we should, therefore, explicitly implement a mechanism to stabilize the orbifold which will then select a solution close to the one with $\sigma = 0$. We will not attempt to do this explicitly but adopt a practical approach and work with background solution for $\sigma = 0$.

We have seen in section 3.3 that a transition from $D = 5$ to $D = 4$ can be implemented "by hand" by decreasing ρ_i to values smaller than one and have discussed qualitatively how this transition can be made to work dynamically. We would now like to realize this transition explicitly, that is, by quantitatively studying the slow-roll time-evolution of the various scalar fields in the model. As explained in section 4 solving the five-dimensional slow-roll equations for the bulk scalar field ϕ is not straightforward and leads to a simple answer only when the bulk and boundary potentials are related in a particular way. For generic potentials and the separating background, $\sigma = 0$, the solution has been explicitly given in appendix D but it turns out to be rather complicated. For the present purpose, we will work with the approximation to this solution discussed in section 4.3 which holds for sufficiently small ρ_i.

In this limit, the appropriate slow-roll conditions are given by eqs. (4.28), (4.29) and (4.30) for ϕ and eqs. (4.10), (4.11) and (4.12) for ϕ_i. The bulk scalar has slow-roll field eq. (4.27) while the brane scalar satisfies (4.8). We remark that a more accurate calculation based on the results of appendix D leads to complicated ρ_i dependent corrections in the bulk slow-roll conditions. Since we plan to study examples where ρ_i is at most $\sim \mathcal{O}(1)$, neglecting these corrections and working with the simple conditions listed above should be a reasonable approximation which gives qualitatively correct results.

Note that the rolling of the bulk scalar field is generally due to the bulk potential as well as due to the boundary potentials, which results in the two terms in the evolution eq. (4.27) for ϕ. Depending on the shape of these potentials this may results in a complicated evolution. To simplify matters, we will focus on cases where $\dot{\phi}$ does not change sign during inflation, so that ϕ does not start to roll up its own potential. Then, the number of e-folds based on the y-averaged scale factor $\bar{\alpha}$ is given by

$$\begin{aligned} N_e &= \int_{\phi_i}^{\phi_f} e^{-\nu}\dot{\alpha}\frac{e^{-\nu}d\phi}{\dot{\phi}} \\ &= \frac{1}{4}\int_{\phi_f}^{\phi_i}\left[\frac{1}{V}\frac{\partial V}{\partial\phi} - \frac{1}{2(V_1+V_2)}\frac{\partial}{\partial\phi}(V_1+V_2)\right]^{-1} e^{-\nu}d\phi\,. \end{aligned} \tag{6.6}$$

Although expressed as an integral over the bulk scalar field, evaluating this integral generally requires solving the evolution eqs. (4.27) and (4.8).

Next, as was considered in section 6, we need to discuss the qualitative evolution of the potentials. Recall that our background solution required V_1+V_2 to be negative. Hence, at least one boundary potential needs to be negative. It is clear that downhill evolution of the corresponding boundary scalar will make this potential even more negative. This is at odds with our goal to evolve from order-one values for $|\rho_i|$ to values smaller than one. A resolution has been proposed in section 5.2. The bulk scalar field, rolling down its own potential, may reverse the trend if the ϕ dependence of the boundary potentials is appropriate. Formally this is governed by eq. (5.4). The first term in the bracket on the r.h.s. describes the change of ρ_i due to the change of ϕ_i and is always positive. The second term governs the change of ρ_i due to ϕ. For a decreasing $|\rho_i|$ we need the r.h.s. of eq. (5.4) to be negative. By inserting the bulk slow-roll field eq. (4.27) we find the conditions stated in section 5.2 that the boundary potentials V_i need to depend on ϕ stronger than $\sqrt{V}$ but weaker than V^2.

A simple class of models (although there are many more examples) satisfying these constraints is specified by a "separating" boundary potential of the form

$$V_i(\phi,\phi_i) = V(\phi)\,\mathrm{U}_i(\phi_i)\,, \tag{6.7}$$

where $\mathrm{U}_i(\phi_i)$ are arbitrary potentials for the boundary scalars. For such potentials we find that eq. (5.4) becomes

$$\frac{e^{-\nu}\dot{\rho}_i}{\rho_i} = -\frac{V}{3\,\mathrm{U}_i\,e^{-\nu}\dot{\alpha}}\left(\frac{\partial\,\mathrm{U}_i}{\partial\phi_i}\right)^2 - \sqrt{\frac{V}{12}}\frac{1}{V^2}\left(\frac{\partial V}{\partial\phi}\right)^2 . \tag{6.8}$$

For expanding branes, $e^{-\nu}\dot{\alpha} > 0$ and hence the first term is positive if $\mathrm{U}_i < 0$. This will drive ρ_i in the wrong direction i.e. towards more negative values. It is clear that this can be overcompensated by the second term if we choose the potential U_i sufficiently flat. This is guaranteed in a particularly simple limiting case, where the boundary potentials are taken to be independent of ϕ_i. It is for this specific case that we would like to analyze two explicit examples. Other more complicated examples can be discussed along similar lines. We consider positive bulk potentials $V = V(\phi) > 0$ and boundary potentials

$$V_i(\phi) = \frac{V(\phi)}{M_i} \tag{6.9}$$

with M_i being constant mass scales. To have $V_1 + V_2 < 0$ we require that

$$\frac{1}{M_1} + \frac{1}{M_2} < 0\,. \tag{6.10}$$

We are interested in two different choices for the bulk potential $V(\phi)$, namely an "inverted quadratic potential" [27] in which to implement new inflation, and a monomial potential for chaotic inflation.

6.1 New inflation model

We consider the potential

$$V = M^2 - \frac{1}{2}m^2\phi^2 + \cdots = M^2(1-x^2) + \cdots , \tag{6.11}$$

where we have introduced two mass scales M and m with $m \ll M$, and the rescaled field

$$x = \frac{m}{\sqrt{2}M}\phi\,. \tag{6.12}$$

As usual higher order terms, indicated by dots in the above equation, have to be added in order to create a minimum. Since we are not interested in the oscillatory regime here, we need not consider these terms explicitly. We start inflation at some value $x = x_I \ll 1$ and roll towards larger values as described by eq. (4.27). Slow-roll breaks down at $x = x_F \simeq 1$. Throughout this range V is positive, as is required for our background solution. The number of e-folds $N_e(x)$ between x_I and x can be obtained from eq. (6.6) and is given by

$$N_e(x) \simeq \frac{M^2}{2m^2}\left(\ln\frac{x}{x_I} - \frac{1}{2}x^2\right) . \tag{6.13}$$

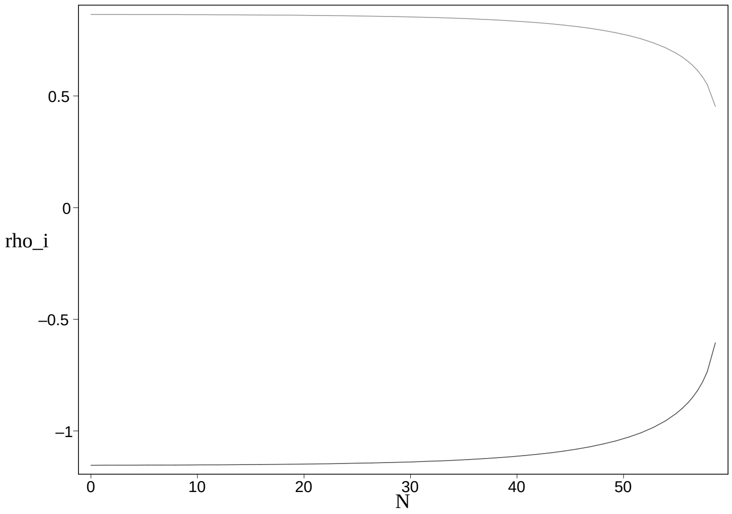

Figure 2: Typical evolution of $\rho_i = V_i/\sqrt{12V}$ for a new inflation bulk potential.

This implies for the total number of e-folds

$$N_e \simeq -\frac{M^2}{2m^2} \ln x_I \,. \tag{6.14}$$

As a function of x, we find for the all-important quantities ρ_i that

$$\rho_i(x) = \frac{M}{\sqrt{12} M_i} \sqrt{1 - x^2} \tag{6.15}$$

which shows that our constraint (6.10) on the mass-scales M_i guarantees a negative value of $\rho_1 + \rho_2$ throughout the evolution, as is required for the background solution. Most importantly, we see that both $|\rho_i|$ decrease in time. This allows a dynamical transition from a genuinely 5D regime at $\rho_i = \mathcal{O}(1)$ to an effective $D = 4$ regime at $\rho_i < 1$. To illustrate this point, we have numerically integrated the evolution equations and, in figure 2, we have plotted ρ_i as a function of the number of e-folds, focusing on the last 60 or so e-folds.

As is expected from the form of the potential, ρ_i stay approximately constant for most of the evolution and only change significantly towards the end of inflation. As a consequence, if one starts with values of $|\rho_i| = \mathcal{O}(1)$ at the beginning of inflation, almost all of inflation takes place in the five-dimensional regime and the transition to the 4D theory only happens during the last few e-folds.

6.2 Chaotic inflation model

Let us now contrast this with what happens for a monomial bulk potential which one expects to be appropriate for chaotic inflation. We consider

$$V = \frac{1}{2} m^2 \phi^2 \,. \tag{6.16}$$

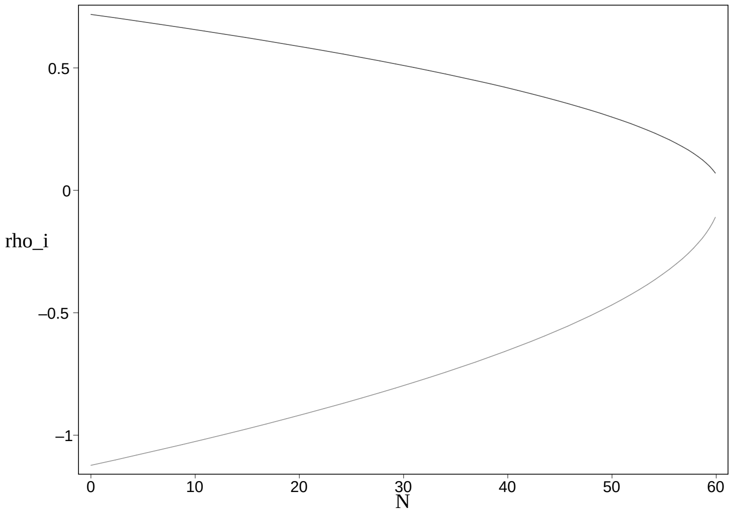

Figure 3: Typical evolution of $\rho_i = V_i/\sqrt{12V}$ for a bulk potential of chaotic type.

We start at some initial $\phi = \phi_I \gg 1$ and, using eq. (4.27) evolve towards smaller ϕ with slow-roll breaking down at $\phi = \phi_F \simeq 2$. The number of e-folds between ϕ_I and ϕ is given by

$$N_e(\phi) = \frac{1}{8}(\phi_I^2 - \phi^2) \tag{6.17}$$

with the total number of e-folds being

$$N_e = \frac{\phi_I^2}{8}\,. \tag{6.18}$$

Furthermore, we find

$$\rho_i(\phi) = \frac{m\phi}{2\sqrt{6}M_i} \tag{6.19}$$

so that $\rho_1 + \rho_2 < 0$ always. Figure 3 shows the corresponding plot for ρ_i as a function of the number of e-folds.

Again, we see that a transition from $D = 5$ to $D = 4$ can be realized dynamically. However, unlike the case for new inflation, here the transition starts early on and the system gradually moves towards the four-dimensional regime during inflation.

One expects that the five-dimensional nature of the inflationary backgrounds just presented will show up in the spectrum of perturbations. Clearly, the structure of $D = 5$ brane-world scalar perturbations and their evolution equations is much richer than in $D = 4$, a fact which supports this view. Moreover, one hopes that the different types of transitions to the four-dimensional regime which we have just encountered will lead to different imprints on the spectrum. A reliable analysis of these questions is beyond the scope of the present paper, and must to be carried out using the formalism for $D = 5$ brane-world perturbations as, for example, developed in refs. [23]–[26].

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A. Classification of dS_5 and AdS_5 solutions in conformal gauge

In this appendix, we will be systematically analyzing the solutions to the equations of motion (2.3)–(2.6) which can be obtained in conformal gauge $\nu = \beta$ and which are part of dS_5 or AdS_5. Here and throughout the paper, we will always take the branes to be located at $y = y_i =$ const where $y_1 = 0$ and $y_2 = R$. This choice, of course, implies that gauge-equivalent solutions to the bulk equations of motion may not be equivalent with respect to these fixed boundary conditions. Therefore, in order to find all possible solutions, we should aim to incorporate co-ordinate reparametrisations into the bulk solution before matching to the boundaries.[6] For now, we will focus on conformal gauge, that is, we incorporate conformal transformations $x_\pm \to \tilde{x}_\pm(x_\pm)$ of the light-cone co-ordinates $x_\pm = \tau \pm y$ into the metric. Later, in appendix C, we will analyze the most general case by allowing arbitrary co-ordinate transformations.

For the metric in conformal gauge, that is,

$$ds^2 = e^{2\beta}(-d\tau^2 + dy^2) + e^{2\alpha}d\mathbf{x}^2 \tag{A.1}$$

and α, β being functions of the light-cone co-ordinates, the bulk equations of motion read

$$\begin{aligned} \partial_\pm^2\alpha - 2\partial_\pm\alpha\partial_\pm\beta + \partial_\pm\alpha^2 &= 0\,, \\ \partial_+\partial_-\alpha + \partial_+\partial_-\beta &= \frac{V}{24}e^{2\beta}\,, \\ \partial_+\partial_-\alpha + 3\partial_+\alpha\partial_-\alpha &= \frac{V}{12}e^{2\beta}\,. \end{aligned} \tag{A.2}$$

These equations can be solved along the lines of ref. [15] where the case $V \leq 0$ has been considered. Following ref. [36], one first notes the existence of a first integral

$$\partial_+\alpha\partial_-\alpha = \left(\frac{V}{48} + Ce^{-2\alpha}\right)e^{2\beta} \tag{A.3}$$

for the system (A.2) with C being an arbitrary integration constant. In this paper, we are focusing on the subclass of bulk solutions with maximal symmetry (that is,

[6] Alternatively, one may incorporate this freedom by working with arbitrary embeddings for the branes, a route that has been taken, for example, in ref. [11, 44]. However, we will not pursue this method in the present paper.

dS_5 or AdS_5 solutions) which are singled out by choosing $C = 0$ as shown in ref. [11]. The most general solution is then given by

$$ds^2 = \frac{1}{(f_+ + f_-)^2} \left\{ -\frac{48}{V} f'_+ f'_- dx_+ dx_- + d\mathbf{x}^2 \right\}, \tag{A.4}$$

where $f_\pm = f_\pm(x_\pm)$ are arbitrary functions of the light-cone co-ordinates and the prime denotes the derivative with respect to the argument. In order to have the correct signature we should restrict these functions so that $\mathrm{sign}(f'_+ f'_-) = \mathrm{sign}(V)$. We should now subject these bulk solutions to the boundary conditions

$$e^{-\beta} \partial_y \alpha \big|_{y=y_i} = e^{-\beta} \partial_y \beta \big|_{y=y_i} = \mp \frac{V_i}{12} . \tag{A.5}$$

This constrains the functions $f_\pm$ to

$$f_+(\tau) = F(\tau) , \qquad f_-(\tau) = \mathrm{sign}(V) \gamma_1^2 F(\tau) + k_1 , \tag{A.6}$$

where F can be expressed in terms of an arbitrary periodic function p satisfying $p(\tau) = p(\tau + 2R)$ as

$$F(t) = e^{-\xi\tau} p(\tau) + \frac{e^{-\xi\tau} - 1}{e^{-2\xi R} - 1} K . \tag{A.7}$$

The coefficients γ_i and ξ are defined by

$$|\gamma_i| = \mp s \rho_i + s_i \sqrt{\mathrm{sign}(V) + \rho_i^2} , \tag{A.8}$$

$$\xi = -\frac{1}{R} \ln \frac{|\gamma_1|}{|\gamma_2|} , \tag{A.9}$$

where the upper (lower) sign in eq. (A.8) refers to the first (second) boundary. Further, K and k_1 are arbitrary constants and we recall that ρ_i is given by

$$\rho_i = \frac{V_i}{\sqrt{12|V|}} . \tag{A.10}$$

Above, we have introduced a number of signs which will turn out to be quite relevant in the following. In detail, these signs are given by

$$s = \mathrm{sign}(f_+ + f_-)\, \mathrm{sign}(f'_-) \tag{A.11}$$

whereas $s_i = \pm 1$ are arbitrary. Normally, one would expect the sign s to be uniquely defined throughout spacetime. Changing s requires the functions $f_+ + f_-$ or f'_- to vanish for certain τ or y. This implies the existence of a co-ordinate singularity corresponding to an infinite scale factor or a horizon, respectively. Even though the curvature scalar remains finite, those solutions may be undesirable due to their exotic properties, such as horizon-separated boundaries or an infinitely large orbifold. Be that as it may, since $f_\pm$ are left- and right-movers, one at least expects that there exists a range of τ for which neither $f_+ + f_-$ nor f'_- change sign as one moves across the orbifold. If necessary, we can focus on this range of τ and then s is unambiguously defined.

We also note that the functions F, as specified by eq. (A.7), can be characterized as solutions of the generalized periodicity condition

$$F(\tau + 2R) = \frac{\gamma_1^2}{\gamma_2^2} F(\tau) + K\,. \tag{A.12}$$

We now wish to investigate some properties of the above solutions. Specifically, we are interested in whether we can find separating solutions i.e. solutions where $\alpha = \alpha_0(\tau) + \alpha_5(y)$ and $\beta = \beta_0(\tau) + \beta_5(y)$) and solutions with a time-independent orbifold i.e. solutions with $\dot{\beta} = 0$). Firstly, we note that for the above class, it can be shown that separability of the solution implies a time-independent orbifold. Secondly, by straightforward computation, one can also demonstrate that a time-independent orbifold requires the condition

$$k_1(1 + \mathrm{sign}(V)\gamma_2^2) = k_2(1 + \mathrm{sign}(V)\gamma_1^2) \tag{A.13}$$

to hold, where we have defined

$$k_1 = \mathrm{sign}(V)\gamma_2^2 K + k_2\,. \tag{A.14}$$

This relation implies the vanishing of the additive constant in $f_+ + f_-$. In view of the structure of the solution (A.4), this is also what one would have intuitively expected as a necessary condition for a time-independent orbifold. Using condition (A.13) we can now compute the sign s for a solution with a time-independent orbifold. Inserting f_+ and f_- into eq. (A.11) we find

$$s = -\,\mathrm{sign}(\xi)\ \mathrm{sign}\left(\mathrm{sign}(V) + \gamma_1^2\right) = -\,\mathrm{sign}(\xi)\ \mathrm{sign}\left(\mathrm{sign}(V) + \gamma_2^2\right)\,. \tag{A.15}$$

We can distinguish four subclasses within the above set of solutions corresponding to the four different combinations of the signs of V and $V_1 + V_2$. It turns out that these subclasses have quite different properties, particularly with respect to separability and existence of static orbifold solutions. We shall therefore discuss them separately.

- $V > 0$: We first note that the signs s_i appearing in eq. (A.8) have to equal $+1$ for $|\gamma_i|$ to be well-defined. As a consequence, we have

  $$|\gamma_i| = \mp s\rho_i + \sqrt{1 + \rho_i^2}\,. \tag{A.16}$$

  Let us now turn to the two sub-cases corresponding to the two different signs of $V_1 + V_2$.

1. $V_1 + V_2 > 0$: In this case, solutions with a time-independent orbifold do not exist. This also implies that there are no separating solutions. The proof is as follows. Using $\rho_1 + \rho_2 > 0$ we conclude from eq. (A.16) that $s|\gamma_2| > s|\gamma_1|$ which implies a positive value of $s\xi$ from eq. (A.9). However, eq. (A.15) tells us that $s\xi$ should be negative for solutions with a static orbifold. It is this sign discrepancy that precludes the existence of static orbifold solutions in the case under discussion.
2. $V_1 + V_2 < 0$: Obviously, an argument analogous to the previous case does not lead to any contradiction for $\rho_1 + \rho_2$ negative. Indeed, separating solutions with constant orbifold exist in this case and explicit examples will be given in the following appendix.

- $V < 0$: In this case, the expression (A.8) for the coefficients γ_i becomes

$$|\gamma_i| = \mp s\rho_i + s_i\sqrt{\rho_i^2 - 1} \tag{A.17}$$

and one immediately concludes that, for the right-hand side to be real, one has to require that

$$|\rho_i| \geq 1\,, \qquad \text{or} \qquad |V_i| \geq 12|V|\,. \tag{A.18}$$

In addition, the right-hand side must be positive which leads to

$$\mathrm{sign}(\rho_1) = -\,\mathrm{sign}(\rho_2) = -s\,. \tag{A.19}$$

In particular, this implies opposite signs for the boundary potentials V_1 and V_2. Note that these are significant constraints on the boundary potentials which arise only if $V < 0$. A further difference from the case $V > 0$ is that the signs s_i appearing in eq. (A.8) are not fixed. We note, however, that they can be expressed as

$$s_i = \mathrm{sign}\left(\gamma_i^2 - 1\right)\,. \tag{A.20}$$

As before, we now distinguish the two signs of $V_1 + V_2$.

1. $V_1 + V_2 > 0$: Similarly to the corresponding $V > 0$ case, no separating solutions or solutions with a static orbifold exist in this case. For the proof, let us consider a solution with a static orbifold. Then, eq. (A.15) combined with eq. (A.20) tells us that the signs s_i have to be the same on both boundaries and are given by $s_i = -s\ \mathrm{sign}(\xi)$. From the definitions of γ_i and ξ, eq. (A.8) and (A.9), we also conclude that $\mathrm{sign}(|\rho_2| - |\rho_1|) = s_i\ \mathrm{sign}(|\gamma_2| - |\gamma_1|) = s_i\ \mathrm{sign}(\xi) = -s$. However, this relation is only compatible with the sign constraints (A.19) on the boundary potentials if $\rho_1 + \rho_2 < 0$.
2. $V_1 + V_2 < 0$: In this case, there is no obstruction to having separating solutions and solutions with a static orbifold and we will present explicit examples in the following appendix.

B. Classification of solutions with constant periodic functions in conformal gauge

We would now like to make the classification of the previous appendix more explicit by focusing on an important subclass of solutions. As we have seen, the general solution obtained in conformal gauge depends, among other things, on an arbitrary periodic function p. Physically, this function represents the freedom to impose arbitrary fluctuations in the orbifold direction as an initial condition. We remark that such initial fluctuations may not be wiped out and, hence, constitute a challenge for inflation particularly with regard to inflationary "no-hair theorems" (see [46] for a recent discussion). Clearly this is a problem present in all higher-dimensional models of inflation but it is particularly acute in brane-world models. While we will not attempt to resolve this problem in the present paper, it is clear that in order to obtain workable cosmological backgrounds, one would like to concentrate on solutions where such fluctuations are small or, equivalently, where the periodic function p is constant. It is this particular subclass that we now wish to analyze in more detail.

Assuming a constant periodic function p in the eq. (A.4), it can be shown that the five-dimensional metric may be written in the form

$$ds^2 = \frac{1}{f^2}\left\{e^{2\beta_0-2\xi\tau}(-d\tau^2+dy^2)+e^{2\alpha_0}d\mathbf{x}^2\right\}. \tag{B.1}$$

The function f is defined by

$$f = \sigma + e^{-\xi\tau}\begin{cases}\cosh(\xi y+\xi_0)\,, & \text{for} \quad V>0\,,\\ \sinh(\xi y+\xi_0)\,, & \text{for} \quad V<0\end{cases} \tag{B.2}$$

and α_0, β_0, ξ, ξ_0 and σ are constants. These constants can be determined, of course, by comparison with the general conformal solution (A.4)–(A.9). However, it is more instructive, and moreover serves as a cross-check of our previous results, to insert the metric (B.1) directly into the equations of motion and the boundary conditions. One then finds that the bulk equations of motion (A.2) are satisfied as long as

$$|\xi|e^{-\beta_0} = \sqrt{\frac{|V|}{12}}\,, \tag{B.3}$$

where the sign of ξ remains undetermined. To incorporate the boundary conditions (A.5) it again proves useful to distinguish four cases as specified by the signs of V and V_1+V_2.

- $V>0$: In this case we define

  $$A_i = \mathrm{arcsinh}(\rho_i) \tag{B.4}$$

  and the boundary conditions are satisfied if

  $$\xi = -\,\mathrm{sign}(f\xi)\frac{A_1+A_2}{R}\,, \qquad \xi_0 = \mathrm{sign}(f\xi)A_1\,. \tag{B.5}$$

The metric (B.1) takes the form

$$ds^2 = \frac{1}{f^2}\left\{\frac{12}{V}\xi^2 e^{-2\xi\tau}(-d\tau^2 + dy^2) + e^{2\alpha_0}d\mathbf{x}^2\right\}, \tag{B.6}$$

where the function f is now given by

$$f = \sigma + e^{-\xi\tau}\cosh\left[(A_1 + A_2)\frac{y}{R} - A_1\right]. \tag{B.7}$$

The constants α_0 and σ remain undetermined. The former can be absorbed into a co-ordinate rescaling and is trivial in this sense. The constant σ, however, cannot be removed in this way and its value is essential for the properties of the solution. Specifically, the solution is separating and has a static orbifold if and only if $\sigma = 0$. In order to see whether such a choice is permissible let us examine the two signs for $V_1 + V_2$ separately.

1. $V_1 + V_2 > 0$: We easily deduce from the first eq. (B.5) that the function f has to be negative. However, from its definition (B.2) it is clear that this is incompatible with setting $\sigma = 0$. In fact, we need $\sigma < 0$. Consequently, no separating solutions or solutions with static orbifold exist in accordance with our conclusion in the previous appendix. As stands, τ takes values in a semi-infinite range. For $\xi > 0$ the solution evolves from a co-ordinate singularity at some finite time corresponding to infinitely separated branes to brane collision as $\tau \to \infty$. The branes scale factors themselves are decreasing, so $\xi > 0$ does not represent inflation. In the opposite case, $\xi < 0$, the models starts out with colliding branes at $\tau \to -\infty$ and ends in a co-ordinate singularity with infinite brane separation at some finite time. Here the branes are inflating. This case is discussed in detail in section 3.2, where f is defined with an extra minus sign so as to make it positive.
2. $V_1 + V_2 < 0$: Equation (B.5) now implies that the function f has to be positive. This, however, places no further constraint on the constant σ. In particular, we are free to set $\sigma = 0$ which leads to a separating solution with static orbifold [9, 15]. For $\sigma \geq 0$ the range of τ is infinite, that is, $\tau \to \pm\infty$ while for $\sigma < 0$ it is restricted to a semi-infinite range with a singularity corresponding to an infinite brane-separation developing at some finite time. These cases are discussed in detail in section 3.3.

- $V < 0$: Defining

$$A_i = \mathrm{arccosh}(|\rho_i|) \tag{B.8}$$

the boundary conditions can be satisfied provided

$$\xi = -\frac{s_1 A_1 + s_2 A_2}{R}, \qquad \xi_0 = s_1 A_1, \tag{B.9}$$

where $s_i = \pm 1$ are, as yet, undetermined signs. For this to provide a well-defined solution to the boundary conditions we need to require in addition that

$$V_1 V_2 < 0\,, \qquad |\rho_i| \geq 1\,. \tag{B.10}$$

These conditions on the boundary potentials are the same we have found for the general conformal solution in the previous appendix, as they should. The metric is then given by

$$ds^2 = \frac{1}{f^2}\left\{\frac{12}{-V}\xi^2 e^{-2\xi\tau}(-d\tau^2 + dy^2) + e^{2\alpha_0}d\mathbf{x}^2\right\} \tag{B.11}$$

with

$$f = \sigma + e^{-\xi\tau}\sinh\left[-(s_1 A_1 + s_2 A_2)\frac{y}{R} + s_1 A_1\right]. \tag{B.12}$$

In fact, our information about the signs of V_i is somewhat more precise, namely

$$\mathrm{sign}(V_1) = -\,\mathrm{sign}(V_2) = \mathrm{sign}(f\xi)\,. \tag{B.13}$$

As with the case where $V > 0$, the constant σ is significant. In particular, $\sigma = 0$ characterizes the separating solutions with static orbifold. To find out whether or not this choice is possible we again distinguish between the two signs of $V_1 + V_2$.

1. $V_1 + V_2 > 0$: No separating solutions or solutions with static orbifold exist in this case. For the proof, let us assume the existence of such a solution with $\sigma = 0$. We focus on times τ where the sign of f in eq. (B.12) does not change across the bulk. Then the signs s_i are fixed to $s_1 = -s_2 = \mathrm{sign}(f)$. From eq. (B.9), one concludes that $\mathrm{sign}(\xi) = \mathrm{sign}(f(|V_2| - |V_1|))$ and, by comparison with eq. (B.13), we learn that $\mathrm{sign}(V_1) = -\,\mathrm{sign}(V_2) = \mathrm{sign}((|V_2| - |V_1|))$. This last equation, however, contradicts our assumption that $V_1 + V_2 > 0$.

   The non-separating solutions with $\sigma \neq 0$ exist with infinite or semi-infinite range of τ depending on the sign of σ and s_i. In either case, the endpoints of the evolution correspond to colliding branes or infinitely separated branes.

2. $V_1 + V_2 < 0$: In this case, there is no obstruction to setting $\sigma = 0$ and, hence, separating solutions with a time-independent orbifold exist. The time-range can be infinite or semi-infinite depending on the values of σ and s_i. For $\sigma \neq 0$ the endpoints of the evolution again correspond to a collapsing or diverging orbifold size. These solutions have been discussed in ref. [9].

C. Classification of solutions in arbitrary gauge

As we have mentioned before, the approach of this paper is to explicitly incorporate co-ordinate reparametrizations into the bulk metric and keep the boundaries at $y =$ const hypersurfaces rather than allowing curved branes. So far, we have focused on conformal transformations and subclasses thereof but it is not clear, *a priori*, that this leads to the most general set of solutions. In this appendix we will therefore allow arbitrary co-ordinate transformations.

The corresponding bulk metric is easily obtained from the one in conformal gauge, eq. (A.4), by transforming to a new set of co-ordinates, $(y^a) = (\tau, y)$ where $a = 0, 5$ and $x_\pm = x_\pm(\tau, y)$. We then have the metric

$$ds^2 = \frac{1}{(F_+ + F_-)^2}\left\{\gamma_{ab}dy^a dy^b + d\mathbf{x}^2\right\}, \tag{C.1}$$

where

$$\gamma_{ab} = -48V\partial_{(a}F_+\partial_{b)}F_- \tag{C.2}$$

and $F_\pm$ are arbitrary functions of τ and y related to their conformal gauge counterparts by

$$F_\pm(\tau, y) = f_\pm(x_\pm(\tau, y)) \,. \tag{C.3}$$

In order for this metric to have to correct signature (with y being the spacelike direction) one should require that

$$\mathrm{sign}(V) = \mathrm{sign}(\dot{F}_+\dot{F}_-) = -\,\mathrm{sign}(F'_+F'_-)\,, \tag{C.4}$$

at least close to the branes.

In this arbitrary gauge, the boundary conditions for the (00) and (ij) components of the metric are still given by eqs. (2.9) and (2.10), while we have to require in addition that g_{05} vanishes at the branes. Interestingly, these conditions can be solved explicitly leading to the following constraints

$$F_-(\tau, y_i) = \mathrm{sign}(V)\gamma_i^2 F_+(\tau, y_i) + k_i\,, \qquad F'_-(\tau, y_i) = -\,\mathrm{sign}(V)\gamma_i^2 F'_+(\tau, y_i) \tag{C.5}$$

on the functions $F_\pm$ at the locations $y = y_i$ of the branes. As before, the co-efficients γ_i are defined by

$$|\gamma_i| = \mp s\rho_i + s_i\sqrt{\rho_i^2 + \mathrm{sign}(V)}\,, \tag{C.6}$$

where $s_i = \pm 1$ are as yet unspecified signs, k_i are constants and the sign s is defined by

$$s = \mathrm{sign}(F_+ + F_-)\ \mathrm{sign}(F'_-)\,. \tag{C.7}$$

Hence, we have found a solution for all functions $F_\pm$ satisfying the boundary conditions (C.5) together with the additional sign constraints (C.4). While these constraints seem relatively weak at first, they may in fact obstruct the existence of a solution under certain conditions. Typically, in these cases one finds that a pair of functions $F_\pm$ which satisfy the boundary conditions on one side, when continued across the bulk in accordance with the constraints (C.4) on $F'_\pm$, can never satisfy the boundary conditions on the opposite side. We will see this explicitly in a moment.

First, however, it is helpful to discuss some geometrical aspects of our solutions following the ideas of ref. [15]. As we will see, the general approach of this appendix is rather well-suited for this purpose. Since our bulk solutions represents a slice of dS_5 or AdS_5 they can be represented as a hypersurface

$$-T^2 + Y^2 + \mathbf{X}^2 + \mathrm{sign}(V) Z^2 = \mathrm{sign}(V)\mathcal{R}^2 \tag{C.8}$$

in a six-dimensional space with co-ordinates $(T, Y, \mathbf{X}, Z)$ and metric

$$ds_6^2 = -dT^2 + dY^2 + d\mathbf{X}^2 + \mathrm{sign}(V) dZ^2 \,. \tag{C.9}$$

Here $\mathcal{R}$ is the radius of the (anti) de Sitter space, given by

$$\mathcal{R}^2 = \frac{1}{k^2} = \frac{12}{|V|} \,. \tag{C.10}$$

It is straightforward to show that the six-dimensional embedding co-ordinates are related to our original five-dimensional ones by

$$\begin{aligned} T &= \mathcal{R}\frac{1 - 4\ \mathrm{sign}(V) F_+ F_- + \frac{\mathbf{x}^2}{\mathcal{R}^2}}{2(F_+ + F_-)} \,, \\ Y &= \mathcal{R}\frac{1 + 4\ \mathrm{sign}(V) F_+ F_- - \frac{\mathbf{x}^2}{\mathcal{R}^2}}{2(F_+ + F_-)} \,, \\ \mathbf{X} &= \frac{\mathbf{x}}{F_+ + F_-} \,, \\ Z &= \mathcal{R}\frac{F_+ - F_-}{F_+ + F_-} \,. \end{aligned} \tag{C.11}$$

In the six-dimensional embedding space, the branes are given as the intersections of certain five-dimensional hyperplanes with the hyperboloid defined by eq. (C.8). Using the relations (C.11) together with the boundary conditions (C.5) on $F_\pm$ we can determine normal vectors $\mathbf{N}_i$ to these hyperplanes. Up to a normalisation constant, we find for these vectors

$$\mathbf{N}_i \sim \left(2k_i, 2k_i, 0, 0, 0, 1 + \mathrm{sign}(V)\gamma_i^2\right) \,. \tag{C.12}$$

These vectors carry information about how the branes are embedded into the $(A)dS_5$ space and, as expected, they depend on the bulk potential V and the respective brane potential V_i through the co-efficients γ_i. However, we also observe that the constants k_i which appear as integration constants in the boundary conditions (C.5), play an important role in specifying this embedding. We can use the normal vectors $\mathbf{N}_i$ to distinguish two qualitatively different types of solution. First, if $\mathbf{N}_1$ and $\mathbf{N}_2$ are not parallel we have a diverging orbifold size in the past (future) and colliding branes in the future (past). In other words, such solutions are characterized by a non-static orbifold. On the other hand, if $\mathbf{N}_1$ and $\mathbf{N}_2$ are parallel the branes never intersect which constitutes a necessary condition for the orbifold to be static. From eq. (C.12) we see that this case is realized precisely when

$$k_1\left(1+\mathrm{sign}(V)\gamma_2^2\right)=k_2\left(1+\mathrm{sign}(V)\gamma_1^2\right) \tag{C.13}$$

which serves as a useful practical condition for a static orbifold.

In order to apply this criterion to our general solution, as before we distinguish the various cases corresponding to the signs of V and V_1+V_2.

- $V>0$: As in the conformal gauge case, the signs s_i in eq. (A.16) must equal $+1$ which leads to

  $$|\gamma_i|=\mp s\rho_i+\sqrt{1+\rho_i^2}\,. \tag{C.14}$$

  For the two signs of V_1+V_2 we find the following:

  1. $V_1+V_2>0$: The previous no-go theorem excluding the existence of solutions with static orbifold can be avoided if the function F'_+ (or F'_-) is allowed to vanish at points in the orbifold. For such solutions, the metric (C.1) develops a co-ordinate singularity leading to a horizon that separates the two boundaries. An example can be obtained by starting with the metric (B.6) with the constant σ set to zero in (B.7). If we introduce a new orbifold co-ordinate z by $g(z)=1/\cosh(\xi y+\xi_0)$, where g is an arbitrary function the transformed metric reads

     $$ds^2=-\frac{12\xi^2}{V}g(z)^2d\tau^2+\frac{12}{V}\frac{g'(z)}{1+g(z)^2}dz^2+e^{2\alpha_0+\xi\tau}g(z)^2d\mathbf{x}^2\,. \tag{C.15}$$

     We find that the boundary conditions (2.9) and (2.10) are solved in these new co-ordinates if

     $$\mathrm{sign}(g(z_i))\sqrt{1+\frac{1}{g(z_i)^2}}=\mp\rho_i\,. \tag{C.16}$$

     If $g(z)\neq 0$ throughout the orbifold, the signs of $g(z_i)$ must be the same and the above conditions cannot be satisfied on both boundaries given that $\rho_1+\rho_2>0$. On the other hand, if $g(z_1)$ and $g(z_2)$ have different signs the obstruction disappears and functions g matching to both boundaries can be found. The function g must then vanish at at least one point in the orbifold. This results in a co-ordinate singularity in the metric (C.15) associated with a horizon separating the two boundaries. An analogous example, for the case of a flat bulk spacetime, has been found in ref. [7].

Having presented the above counter-example, let us now exclude co-ordinate singularities and assume that the functions F'_+, F'_- as well as $F_+ + F_-$ are non-vanishing throughout the spacetime. Under this additional assumption, it can again be shown that static orbifold solutions do not exist. For the proof, we have to analyze the boundary conditions on $F_\pm$ given by eq. (C.5). A useful observation which simplifies the argument is that these functions can be shifted by a constant c according to $F_+ \to F_+ - c$ and $F_- \to F_- + c$ without affecting the bulk metric. The boundary conditions will, of course, be modified. Let us choose the constant c to be $c = k_1/(1+\gamma_1^2)$. After the shift the additive constant in the condition on the first boundary will be removed. If, in addition, we assume a static orbifold, we learn from eq. (C.13) that the same is true for the second boundary. Then, the new boundary conditions are simply

$$F_-(\tau, y_i) = \gamma_i^2 F_+(\tau, y_i)\,. \tag{C.17}$$

These equations, together with our initial assumption about the absence of co-ordinate singularities implies that the signs of $F_+(\tau, y_i)$ and $F_-(\tau, y_j)$ must be the same for all $i, j = 1, 2$. In addition, when continued across the orbifold the functions $F_\pm$ have to respect the sign constraint (C.7). This leads to $s|F_-(\tau, y_1)| > s|F_-(\tau, y_2)|$ and $s|F_+(\tau, y_1)| < s|F_+(\tau, y_2)|$. From eq. (C.17) this means that we can only find functions $F_\pm$ matching both boundaries if $s\gamma_1^2 > s\gamma_2^2$. However, this contradicts the definition for γ_i, eq. (C.16) which implies that $s\gamma_1^2 < s\gamma_2^2$ as long as $\rho_1 + \rho_2 > 0$.

2. $V_1 + V_2 < 0$: As we have already seen, static orbifold solutions exist in this case.

- $V < 0$: The coefficients γ_i are given by

$$|\gamma_i| = \mp s\rho_i + s_i\sqrt{\rho_i^2 - 1} \tag{C.18}$$

and, as in the more special cases, are only well-defined if the potentials obey

$$|V_i| \geq 12|V|\,, \qquad V_1 V_2 < 0\,. \tag{C.19}$$

Hence, these significant constraints on the potentials persist even in the most general case. If they are violated no solutions exist.

1. $V_1 + V_2 > 0$: Arguments analogous to the ones in the $V > 0$ case do not apply any longer, essentially due to the additional sign freedom s_i in eq. (C.16) which, unlike in the case of conformal gauge, can no longer be fixed. We expect static orbifold solutions to exist, even if the metric is required to be free of co-ordinate singularities.
2. $V_1 + V_2 < 0$: As previously shown, static orbifold solutions exist.

D. Bulk slow-roll for the static orbifold solution

In this appendix, we will solve the bulk eq. (2.7) for ϕ for a static orbifold solution with $V > 0$ and $V_1 + V_2 < 0$. To do this, we follow a procedure similar to the one explained in section 4. We start by introducing a new variable $\mathcal{T} = \mathcal{T}(\tau, t)$ with

$$\frac{d\phi}{dT} = -3\frac{\partial_\phi V}{V}\,. \tag{D.1}$$

We are now searching for solutions where the bulk scalar field depends on τ and y only through this new variable, that is, $\phi = \phi(\mathcal{T}(\tau, y))$. It is straightforward to show from the ϕ equation of motion (4.4) that, with the metric in conformal gauge, $\mathcal{T}$ then satisfies the differential equation

$$\ddot{\mathcal{T}} - T'' + 3\dot{\alpha}\dot{\mathcal{T}} - 3\alpha'\mathcal{T}' = \frac{1}{3}e^{2\beta}V \tag{D.2}$$

together with the boundary conditions

$$[\mathcal{T}']_{y=y_i} = \mp\left[e^{\beta}\frac{\partial_\phi V_i}{\partial_\phi V}V\right]_{y=y_i} \tag{D.3}$$

provided that

$$12\left(\frac{\partial_\phi^2 V\,\partial_\phi V}{V^2} - \frac{(\partial_\phi V)^3}{V^3}\right)(\dot{\mathcal{T}}^2 - \mathcal{T}'^2) \ll e^{2\beta}\partial_\phi V \tag{D.4}$$

holds. This inequality will eventually lead to slow-roll conditions once an explicit result for $\mathcal{T}$ is inserted. We observe that, from eq. (D.1), ϕ should have a weak dependence on $\mathcal{T}$ for flat bulk potentials V. Finding an exact solution for $\mathcal{T}$ by solving eq. (D.2) is quite difficult since the background quantities α and β contain the various potentials which have a time-variation through slow-roll of the scalar fields. However, our strategy will be to solve eq. (D.2) by assuming that the potentials are effectively constant. This approximation neglects higher-order contributions to the slow-roll evolution of ϕ which, to leading order is governed by (D.1). This, of course, means that our final solution for ϕ is only valid locally and describes the τ and y variation of ϕ around a given set of potential values.

As we saw in section 4, even in this approximation, solutions for $\mathcal{T}$ satisfying the boundary conditions (D.3) are not easily found. However, for the $\sigma = 0$ separating background (3.17) we can start with

$$\mathcal{T} = \alpha + \mathcal{T}_0(\tau) + \mathcal{T}_5(y)\,, \tag{D.5}$$

where the first term, α, is a special inhomogeneous solution and the other terms represent a separation Ansatz for the homogeneous part of (D.2). The fact that $\mathcal{T} = \alpha$ is a special solution of eq. (D.2) can be shown rather generally by using the background equations of motion (2.3)–(2.6) and their first integral (4.9). The existence of this simple solution is of course the prime motivation for introducing the variable $\mathcal{T}$. For our specific separating background α takes the form

$$\alpha = \alpha_0 + \xi\tau - \ln\cosh(\xi y + \xi_0)\,. \tag{D.6}$$

The most general separating solution for the homogeneous part is given by

$$\begin{aligned}\mathcal{T}_0 &= C\xi\tau + Ae^{-3\xi\tau}\,,\\ \mathcal{T}_5 &= BH(\xi y + \xi_0) + CI(\xi y + \xi_0)\,,\end{aligned} \tag{D.7}$$

where A, B, C are arbitrary integration constants and the functions H and I are defined by

$$\begin{aligned}H(z) &= (3 + \sinh^2 z)\sinh z\,,\\ I(z) &= \ln\cosh z + \frac{1}{2}\cosh^2 z + \sinh z\cosh^2 z \arctan e^z + 2\sinh z \arctan\, e^z\,.\end{aligned} \tag{D.8}$$

We remark that I is a special solution of

$$\frac{d^2 I}{dz^2} - 3\tanh z\frac{dI}{dz} = 3 \tag{D.9}$$

while H satisfies the homogeneous part of this equation. We now have to implement the boundary conditions (D.3) which fixes B and C to

$$\begin{aligned}B &= B_1\left(1 - 2\frac{\delta_1}{\delta}\right) + B_2\left(1 - 2\frac{\delta_2}{\delta}\right),\\ C &= C_1\left(1 - 2\frac{\delta_1}{\delta}\right) + C_2\left(1 - 2\frac{\delta_2}{\delta}\right),\end{aligned} \tag{D.10}$$

where we recall that

$$\delta_i = \frac{\partial_\phi V_i}{V_i}\,,\qquad \delta = \frac{\partial_\phi V}{V}\,. \tag{D.11}$$

Further B_i and C_i are complicated functions of ρ_i given by

$$\begin{aligned}B_1 &= D^{-1}\sqrt{1+\rho_2^2}\, I'(-A_2)\rho_1\,,\\ B_2 &= D^{-1}\sqrt{1+\rho_1^2}\, I'(A_1)\rho_2\,,\\ C_1 &= -3\, D^{-1}\left(1+\rho_2^2\right)^2\rho_1\,,\\ C_2 &= -3\, D^{-1}\left(1+\rho_1^2\right)^2\rho_2\,,\\ D &= 3\sqrt{1+\rho_1^2}\sqrt{1+\rho_2^2}\left[\left(1+\rho_1^2\right)^{3/2}\, I'(-A_2) - \left(1+\rho_2^2\right)^{3/2}\, I'(A_1)\right].\end{aligned} \tag{D.12}$$

This result can now be used to derive the various slow-roll conditions for the background solution under consideration by inserting $\mathcal{T}$ into the inequalities (4.1), (4.2), (4.3) and (D.4). Further conditions arise by inserting into the equation of motion (D.2) for $\mathcal{T}$ and by requiring that additional terms proportional to time derivatives of potentials are indeed small, as we have assumed. Clearly, this will lead to very complicated slow-roll conditions, in general. The above solution significantly simplifies if expanded to lowest order in ρ_i. Then we arrive at the ϕ evolution eq. (4.27) and the slow-roll conditions (4.28),(4.29) and (4.30) which we have used to analyze the specific examples in section 6.

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